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A COMPARISON OF SOLAR ATMOSPHERIC MODELS BY STARK EFFECT IN H-GAMMA : NITROGEN ABUNDANCE

D. Mugglestone

(Received 1959 September 7)

Summary

Three solar atmospheric models are considered, i.e., the models of Claas, Vitense and Swihart. The criterion of reliability adopted here is that the models should correctly predict the observed intensities in the far wing of the H-gamma absorption line. An improved theory of Stark broadening is used, due to Dr A. C. Kolb, where the ions are treated according to the usual Holtsmark statistical theory and also the effect of the electrons, and their finite velocities, is included. It is found that the electronic and ionic contributions to the absorption coefficient are of the same order of magnitude, in the far wing, and therefore the electronic effect should not be neglected. Results indicate that, of the models considered, the Swihart model is to be preferred.

In a previous paper (9) the nitrogen abundance in the solar atmosphere was determined, using each of the above models, by a method dependent only upon the physical characteristics of the chosen atmospheric model. Placing now additional weight on the results obtained using the Swihart model, the most probable value of the nitrogen abundance is found to be log $A_N = 7{\cdot}86$ (on the basis log $A_H = 12{\cdot}00$).

Introduction.—In an earlier paper (**9**), concerned with the determination of solar atmospheric abundances, a method of determination was developed, the Planckian gradient method, which was based entirely upon the physical characteristics of the assumed theoretical model atmosphere. In order to examine the numerical predictions of the method, and particularly to illustrate the divergencies when compared with the method of weighting functions, an element of high ionization potential was considered (nitrogen, I.P. 14·54 volts). It was also of interest to examine the sensitivity of the method to the particular atmospheric model and therefore three models were chosen: (i) the model adopted by Claas (**2**), which is referred to as the "Claas model", (ii) the model of Vitense (**14**) and (iii) the Swihart model (**12**). The method of derivation, and the detailed models themselves, are described in the aforementioned paper. It was found that the derived nitrogen abundance was moderately insensitive to the adopted atmospheric model. In obtaining an estimate of the most probable value of the nitrogen abundance, however, it is necessary to obtain some measure of the reliability of each of the atmospheric models. It is the purpose of the present paper to obtain such a measure.

The criterion of reliability adopted here is the correct prediction, by the model atmosphere, of the observed line profile in the far wing of the Hγ absorption line ($\lambda = 4340{\cdot}5$ A). The Balmer lines are chosen since they are very strong and can be measured accurately in the far wing; in addition the excitation potential

of the lower level (10·20 volts) is very close to the excitation potential of the nitrogen lines (10·33 volts). A consequence of this latter point is that the Balmer lines have approximately the same temperature sensitivity as the nitrogen lines and therefore originate in the same regions of the model atmosphere.

Of the various physical mechanisms producing a broadening of spectral lines, e.g., natural broadening, Doppler effect, collision damping, Stark effect, etc., it is found that, in the far wing of strong lines, the contribution to the absorption produced by the former effects is negligible and line broadening is almost entirely due to the Stark effect. We wish then to predict the line profile in the far wing of Hγ, i.e., sufficiently far from the line centre that effects other than the Stark effect may be neglected but not so very far in the wing that observational measurements become uncertain.

1. *Kolb's theory of the first order Stark effect.*—Most theoretical considerations of the problem of Stark broadening, outside the narrow Doppler core, have been based upon the Holtsmark (4) statistical theory where the radiating atom is subjected to long-range coulomb forces due to neighbouring ions which produce an "average" field. The slowly moving ions are treated as though stationary during the times of interest and the field produced is then dependent only upon the ion density. Kolb (5), however, has refined the theory and takes into account not only the ions but also the effect of the electrons and their finite velocities. A detailed account of Kolb's theory has recently been published by Aller and Jugaku (1); in the present paper we merely restate Kolb's results for the far wing of the Hγ line while retaining Kolb's original notation.

In Kolb's theory the ion broadening is treated according to the usual Holtsmark theory, where the ions produce an average field at the radiating hydrogen atom which produces a Stark resolution of the degenerate hydrogen substates. The high velocity electrons broaden these static field states since (*a*) they produce phase shifts in the emitted wave train according to the "weak collision" theory, and (*b*) they induce transitions between the degenerate states (the non-adiabatic effect). The total absorption coefficient will be proportional to the sum of the individual broadened Stark components weighted in accordance with their theoretical intensity. In the present theory for Hγ the non-adiabatic effects have been taken into account in only a very approximate way, i.e., Kolb has calculated the non-adiabatic effects accurately only for Lyman-α (where he found that the adiabatic contribution was approximately half that of the non-adiabatic effects) and he has tentatively assumed that approximately the same relative contribution holds for the Balmer lines.

If Δw is the displacement, in circular frequency units, of the Stark component from the line centre, i.e.,

$$\Delta w = 2\pi\Delta\nu = 2\pi c\frac{\Delta\lambda}{\lambda^2} \tag{1}$$

(for Hγ, $\lambda = 4340$ A, $\Delta w = \Delta\lambda \,.\, 10^{12}$) we may define a parameter

$$\beta = \frac{\Delta w}{\bar{\gamma}_k} \tag{2}$$

as a measure of the ion field, where $\bar{\gamma}_k$ is the ion half-width for the Holtsmark average field strength:

$$\bar{\gamma}_k = 4{\cdot}524 n_i X_k. \tag{3}$$

Here n_i is the ion density (which is taken to be equal to the electron density n_e) and X_k is an integer which is characteristic of the displacement of the kth Stark component. Similarly we define a parameter

$$x = \frac{\Delta w}{\gamma_k} \tag{4}$$

as a measure of the electron field, where γ_k is the half-width for electron broadening. Kolb finds

$$\gamma_k = 1{\cdot}22 \times 10^{-4} X_k^2 \frac{n_e}{\sqrt{T}} F_k, \tag{5}$$

where F_k is a function of the parameter δ_k and

$$\delta_k = 1{\cdot}146 \times 10^{-6} X_k \frac{\sqrt{n_e}}{T}, \tag{6}$$

actually

$$F_k = 0{\cdot}2124 - 0{\cdot}383 \log \delta_k. \tag{7}$$

A factor 2 has been included in equation (6) to allow, in an approximate way, for the non-adiabatic effects which were calculated accurately only for the Lyman series.

The line absorption coefficient per absorbing atom for each Stark component k is defined as

$$\alpha_k(\Delta w)\, d(\Delta w), \tag{8}$$

with the normalization condition

$$\sum_k \int_0^\infty \alpha_k(\Delta w)\, d(\Delta w) = 2\pi \left(\frac{\pi e^2}{mc}\right) f_{\mathrm{D}} \tag{9}$$

where f_{D} is the total oscillator strength of the displaced components (for $H\gamma$, $f_{\mathrm{D}} = 0{\cdot}0377$). Weighting each component by its theoretical intensity Kolb finds

$$\alpha_k(\Delta w)\, d(\Delta w) = 2\pi \left(\frac{\pi e^2}{mc}\right) f_{\mathrm{D}} \frac{2}{\pi} \frac{f_k}{\sum f_k} \cdot \frac{1}{\bar{\gamma}_k} I_k(\Delta w)\, d(\Delta w), \tag{10}$$

where f_k is proportional to the f-value of the kth component, and $I_k(\Delta w)$, the intensity of the individual Stark components, is given by

$$I_k(\Delta w) = \left(\frac{x}{\beta}\right) \frac{x^2}{(x^2+1)^2} + \frac{2{\cdot}35 A_2(x)}{\beta^{5/2} \cdot 2} + \dots . \tag{11}$$

Here the parameter $A_2(x)$ has the value

$$A_2(x) = \frac{1}{0{\cdot}7071} \frac{\sin(7/2 \tan^{-1} x)}{(1 + 1/x^2)^{7/4}} \tag{12}$$

which has been tabulated by Kolb (5). In order to obtain the total absorption coefficient at any value of Δw we must now sum over all Stark components:

$$\alpha(\Delta w) = \sum_k \alpha_k(\Delta w) = \left[2\pi \left(\frac{\pi e^2}{mc}\right) f_{\mathrm{D}} \frac{2}{\pi}\right] \sum_k \frac{f_k}{\sum f_k} \cdot \frac{1}{\bar{\gamma}_k} I_k(\Delta w). \tag{13}$$

Introducing the numerical values of the constants and writing $s_k = f_k/\sum f_k$ we have (noting equations (2) and (11))

$$\begin{aligned}
\alpha(\Delta w) &= 0{\cdot}00400 \sum_k s_k \frac{1}{\bar{\gamma}_k} I_k(\Delta w) \\
&= 0{\cdot}00400 \sum_k s_k \left[\frac{x}{(\beta\bar{\gamma}_k)} \frac{x^2}{(x^2+1)^2} + \frac{1{\cdot}175 A_2(x)}{(\beta\bar{\gamma}_k)\beta^{3/2}} + \dots \right] \\
&= \frac{0{\cdot}00400}{\Delta w} \sum_k s_k \left[\frac{x^3}{(x^2+1)^2} + \frac{1{\cdot}175 A_2(x)}{\Delta w^{3/2}} \left(\frac{\bar{\gamma}_k}{X_k}\right)^{3/2} X_k^{3/2} + \dots \right] \\
&= \frac{0{\cdot}00400}{\Delta w} \sum_k s_k A_1(x) + \frac{0{\cdot}00470}{\Delta w^{5/2}} \left(\frac{\bar{\gamma}_k}{X_k}\right)^{3/2} \sum_k s_k X_k^{3/2} A_2(x) + \dots . \qquad (14)
\end{aligned}$$

It is found that the higher terms are unimportant in the far wing, i.e., when Δw is large compared with the half-width for electron broadening $\left(x = \dfrac{\Delta w}{\gamma_k} \gg 1\right)$. When this condition holds

$$A_1(x) = \frac{x^3}{(x^2+1)^2} \to \frac{1}{x} \qquad (15)$$

and the equation, valid for the far wing, becomes finally

$$\alpha(\Delta w) = \frac{0{\cdot}00400}{\Delta w} \sum_k s_k \frac{1}{x} + \frac{0{\cdot}00470}{\Delta w^{5/2}} \left(\frac{\bar{\gamma}_k}{X_k}\right)^{3/2} \sum_k s_k X_k^{3/2} A_2(x). \qquad (16)$$

The second term in equation (16) is the usual Holtsmark expression (see, e.g., Aller and Jugaku (**1**)) with the exception of the factor $A_2(x)$ which, however, is of the order of unity for large values of x. It will also be noted that the absorption coefficient is dependent upon both density and temperature, unlike the result obtained by the Holtsmark theory which is temperature independent.

The Kolb theory, essentially in the above form, has been applied to A star atmospheres by Osawa (**10**) and to B star atmospheres by Elste, Jugaku and Aller (**3**); these investigators conclude that the Kolb theory predicts the observed line profiles more closely than does the Holtsmark theory. More recently the theory has been revised and generalized by Kolb and Griem (**6**) in a form applicable to the line centre as well as to the wing. Kolb (**7**) points out that the revised theory indicates that the shape of the distribution is the same as that given by the above theory, although the required correction to the Holtsmark theory may be slightly larger than indicated above.

2. *Calculation of the line absorption coefficient l_ν.*—According to Kolb the f-numbers for the Stark components of Hγ are

X_k	f_k	X_k	f_k	X_k	f_k
0	1416	10	1760	20	80
2	156	12	166	22	7
3	922	13	1664		
5	192	15	1152	$\sum f_k = 9016$	
7	116	17	52		
8	15	18	1318		

the factor s_k may then be calculated directly.

The calculations are carried out in the following steps: The atmospheric model provides details of the electron pressure P_e (and hence n_e since $P_e = n_e kT$)

and temperature as a function of optical depth, i.e., $\tau=\tau(P_e, T)$. For given values of Δw and τ we may compute:

(i) δ_k from equation (6) and F_k from equation (7) for each Stark component k;
(ii) $X_k^2F_k$ and hence γ_k from equation (5);
(iii) $\bar{\gamma}_k/X_k$ from equation (3);
(iv) the factors, variable with k, of the 1st and 2nd terms of equation (16) for each Stark component k;
(v) the sum of (iv) over all Stark components;
(vi) α in accordance with equation (16).

This process was repeated at moderately regular intervals of optical depth, and at three distances from the line centre $\Delta\lambda=5, 6, 7$ A corresponding with $\Delta w=5, 6, 7, \times 10^{12}$ (see equation (1)), for each of the three model atmospheres. It is found that the ionic and electronic contributions to the absorption coefficient are of the same order of magnitude and therefore, in the far wing, the electronic component should not be neglected.

The line absorption coefficient per absorbing atom, α_ν, has thus been obtained in accordance with the Kolb theory and the transition to the mass absorption coefficient l_ν may now be made by the relationship

$$l_\nu=\frac{N_i}{N_{\mathrm{H}}m_{\mathrm{H}}}\alpha_\nu, \tag{17}$$

where N_i is the number of absorbing atoms per cm^3, and $N_{\mathrm{H}}m_{\mathrm{H}}$ gm cm^{-3} is the assumed density of the solar atmosphere (due to the overwhelming abundance of hydrogen). Introducing Boltzmann's equation we have

$$\frac{N_i}{N_{\mathrm{H}}}=\frac{g_i\,.\,10^{-V\theta}}{u}, \qquad \theta=\frac{5040}{T} \tag{18}$$

where g_i is the statistical weight $(2n^2)$ of the lower level concerned in the transition and V is the excitation potential of this level ($V=10{\cdot}196$ volts). The partition function for hydrogen has been calculated by Claas (2), i.e., $u=2{\cdot}0$. Equation (18) then becomes

$$l_\nu=\frac{4\,.\,10^{-V\theta}}{m_{\mathrm{H}}}\alpha_\nu. \tag{19}$$

The line absorption coefficient l_ν can therefore be obtained for the optical depths $\tau_\nu=\tau_\nu(\theta)$ at which α_ν has been previously calculated and graphical interpolation provides l_ν at regular intervals of τ_ν.

3. *Prediction of the wing of the Hγ absorption line*.—The intensity dip in the continuum, at the centre of the solar disk, for radiation of frequency ν is

$$\Delta I_\nu=\int_0^\infty \frac{dB_\nu}{d\tau_\nu}\mathrm{e}^{-\tau_\nu}\left[\int_0^{\tau_\nu}\frac{l_\nu}{k_\nu}\,d\tau_\nu\right]d\tau_\nu \tag{20}$$

for a line formed by pure absorption, where $B_\nu(\theta)$ is the Planck function and k_ν is the continuous absorption coefficient. Equation (20), used by the author for abundance determinations in a previous paper (9), is found to be unsuitable for the present purpose since the ratio l_ν/k_ν is very small, except at large optical depths, and the integration $\int_0^{\tau_\nu} l_\nu/k_\nu\,d\tau_\nu$ is therefore rather uncertain. An alter-

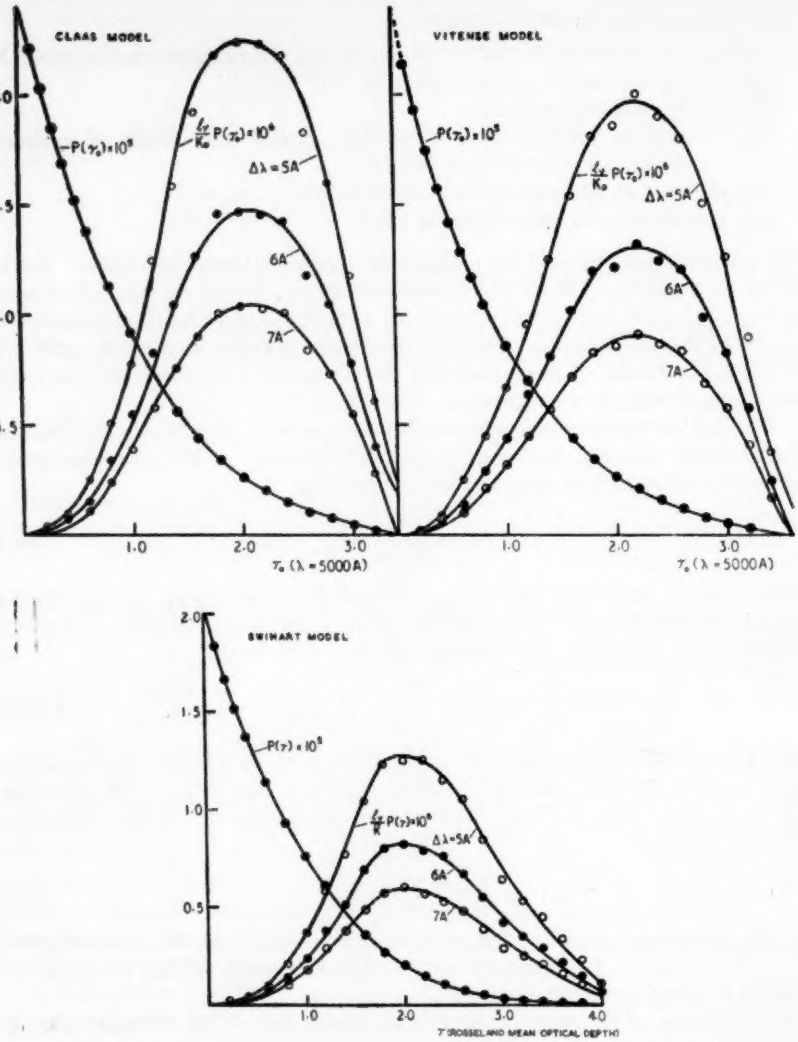

FIG. 1.—*Contribution to absorption as a function of optical depth for the wing of H-gamma (5, 6 and 7A from line centre).*

native form of this equation (see Mugglestone (**9**) equations (6.4) and (6.5) there) previously obtained by Unsöld (**13**) is preferred, i.e.,

$$\Delta I_\nu = \int_0^\infty \left[\int_{\tau_\nu}^\infty B_\nu e^{-\tau_\nu}\, d\tau_\nu - B_\nu e^{-\tau_\nu} \right] \frac{l_\nu}{k_\nu}\, d\tau_\nu \tag{21}$$

which was numerically evaluated in the following manner:—

$$\text{(i)}\ P(\tau_0) = \int_{\tau_0}^\infty B_\nu e^{-\tau_\nu} \frac{k_\nu}{k_0}\, d\tau_0 - B_\nu e^{-\tau_\nu}.$$

The ratio of absorption coefficients k_ν ($\lambda = 4340$ A) to k_0 (absorption coefficient at standard wave-length $\lambda = 5000$ A) were obtained from tables supplied by Dr B. Strömgren (unpublished), hence the relationship between the corresponding optical depths could be calculated from

$$\tau_\nu = \int_0^{\tau_0} \frac{k_\nu}{k_0} d\tau_0 \tag{22}$$

at regular intervals of optical depth τ_0. The particular atmospheric model adopted provides $\tau_0 = \tau_0(\theta)$, and since for a given wave-length ($\lambda = 4340$ A) the Planck function is a function of optical depth only, i.e. $B_\nu = B_\nu(\theta)$, $B_\nu e^{-\tau_\nu} k_\nu/k_0$ may be determined for each value of τ_0. The required integrations

$$\int_0^\infty B_\nu e^{-\tau_\nu} \frac{k_\nu}{k_0} d\tau_0 - \int_0^{\tau_0} B_\nu e^{-\tau_\nu} \frac{k_\nu}{k_0} d\tau_0$$

were calculated in accordance with the Gaussian quadrature formulae and hence the function $P(\tau_0)$ is obtained at values of τ_0 with regular interval. (See Fig. 1.)

(ii) $\Delta I_\nu = \int_0^\infty \frac{l_\nu}{k_0} P(\tau_0) d\tau_0$. Values of the continuous absorption coefficient $k_0 = k_0(\theta, P_e)$ are available from tables of $k_0/\bar{k} = k_0/\bar{k}(\theta, P_e)$ (Strömgren, unpublished) and $\bar{k} = \bar{k}(\theta, P_e)$ (Strömgren (**11**)), and the line absorption coefficient l_ν is calculated directly from equation (19) utilising the Stark effect results obtained from Kolb's theory. The results obtained for the integrand $l_\nu/k_0 . P(\tau_0)$ are shown graphically in Fig. 1 and the resulting ΔI_ν, calculated in accordance with the Gaussian quadrature formula and expressed as a percentage of the intensity in the continuum, are given in Table I, i.e. the intensity in the continuum I_ν^c, which is given by

$$I_\nu^c = \int_0^\infty B_\nu e^{-\tau_\nu} \frac{k_\nu}{k_0} d\tau_0 \tag{23}$$

has been computed in section (i) and hence the predicted $\Delta I_\nu / I_\nu^c$ may be compared with the observed intensities in the wing of the Hγ line as measured in the Utrecht Photometric Atlas (**8**) (see also Fig. (2)).

Table I

Predicted dip in the continuum ΔI_ν expressed as a percentage of the continuum intensity (i.e. $\Delta I_\nu / I_\nu^c$) and compared with the observed intensities in the wing of the H-gamma line.

Model atmosphere	$\Delta\lambda = 5$ A per cent	$\Delta\lambda = 6$ A per cent	$\Delta\lambda = 7$ A per cent
Claas model	14·1	9·3	6·6
Vitense model	13·6	8·9	6·3
Swihart model	10·5	6·8	4·9
Observed	8·8	6·3	4·6

4. *Conclusions.*—The results obtained in the above calculations, together with the results of the recently revised theory of Kolb and Griem (i.e., that the electronic contribution to the absorption coefficient should be further increased, thereby increasing the predicted absorption in the far-wing), clearly indicate that, on the basis of this criterion of reliability, the Swihart model should be preferred. It is clear however that this preference should be applied only for absorption lines formed mainly in the lower layers of the atmosphere and for

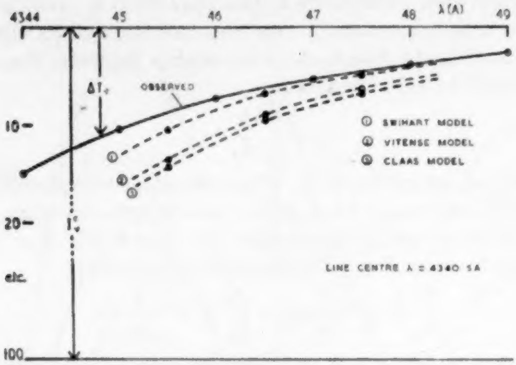

FIG. 2.—*Wing of H-gamma. Intensities predicted by theoretical models compared with observed values.*

which equations (20) and (21) are applicable: i.e. weak lines (or the far wings of strong lines) of high excitation potential.

With regard to the nitrogen abundance in the solar atmosphere a previous paper (9) established the following results: $\log A_N = 7\cdot71$ (Claas model), $\log A_N = 7\cdot81$ (Vitense model) and $\log A_N = 7\cdot93$ (Swihart model). In view of the results of the present paper it has been decided, somewhat arbitrarily, to assign triple weight to the results obtained using the Swihart model and single weight to the Vitense and Claas models. On this basis the most probable value of the nitrogen abundance is

$$\log A_N = 7\cdot86 \quad (\log A_H = 12\cdot00).$$

The author would like to thank Dr A. C. Kolb of the United States Naval Research Laboratory, Washington, for advance information with respect to his theory of Stark broadening, and for private correspondence entered into in this regard.

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1959 *August*.

References

(1) Aller, L. H., and Jugaku, J., *Ap. J.*, **128**, 616, 1958.
(2) Claas, W. J., *Rech. Obs. Utrecht*, **12**, 13, 1951.
(3) Elste, G., Jugaku, J., and Aller, L. H., *Pub. A.S.P.*, **68**, 23, 1956.
(4) Holtsmark, J., *Physik. Zs.*, **25**, 73, 1924.
(5) Kolb, A. C., Thesis, University of Michigan, 1957.
(6) Kolb, A. C., and Griem, H., *Phys. Rev.*, **111**, 514, 1958.
(7) Kolb, A. C., private communication.
(8) Minnaert, M., Mulders, G. F. W., and Houtgast, J., *Photometric Atlas of the Solar Spectrum*, Utrecht, 1940.
(9) Mugglestone, D., *M.N.*, **118**, 432, 1958.
(10) Osawa, K., *Ap. J.*, **123**, 513, 1956.
(11) Strömgren, B., *Pub. Medd. København Obs.*, **138**, 1944.
(12) Swihart, T. L., *Ap. J.*, **123**, 143, 1956.
(13) Unsöld, A., *Zs. f. Astrophys.*, **4**, 339, 1932.
(14) Vitense, E., *Zs. f. Astrophys.*, **34**, 209, 1954.

TRANSITION PROBABILITIES FOR FORBIDDEN LINES OF Ne IV

R. H. Garstang

(Received 1959 September 18)

Summary

Transition probabilities have been computed for magnetic dipole and electric quadrupole radiation for transitions between the levels of the $2p^3$ configuration of Ne IV. The results of a calculation of the relative intensities of lines of the ^{2}P–^{2}D multiplet are compared with observation.

The forbidden lines of neon are of particular interest in planetary nebulae, not least because lines of three consecutive stages of ionization (Ne III, Ne IV and Ne V) are frequently observed in the same object. Published lists of transition probabilities usually include a few lines of [Ne IV], but for calculations on the physical state of the nebulae information is needed on all transitions in [Ne IV], whether in the accessible spectrum or not. The purpose of this paper is to provide such a complete list. The calculations involved are similar to those for [O II], described by Aller, Ufford and Van Vleck (**2**) and by Seaton and Osterbrock (**12**). We use the same notation as in their papers.

The spin–orbit parameter ζ.—This was estimated by interpolating $\zeta^{1/4}$ between Ne III and Ne V, Na IV and F IV, and Mg V and O III, the values of ζ used for all these ions being those of Garstang (**5**). The values of ζ obtained by the three methods agree very closely, and the final value adopted for Ne IV is $\zeta = 692$ cm^{-1}.

The spin–spin parameter η.—Interpolation of $\eta^{1/3}$ between Ne III and Ne V, Na IV and F IV, and Mg V and O III gives, respectively, $\eta = 4{\cdot}92$, $4{\cdot}28$, $4{\cdot}17$ cm^{-1}. Examination of the values of η for all these ions (**5**) shows that the value of $\eta^{1/3}$ for Ne III lies off the best linear law of $\eta^{1/3}$ against Z for the $2p^4$ configuration. If the value of $\eta^{1/3}$ is smoothed before interpolation between Ne III and Ne V a value $\eta = 4{\cdot}49$ cm^{-1} is obtained for Ne IV. As a further check η was computed from the self-consistent field with exchange wave function of Froese and Hartree (**4**), the result being $\eta = 4{\cdot}48$ cm^{-1}. Taking all these values into consideration $\eta = 4{\cdot}40$ cm^{-1} was adopted for Ne IV.

The term separations (PS), (PD), (DS).—Using energy levels given by Moore (**8**, with additions in Appendix to Volume III based on Bowen (**3**)), we adopt (PS) $= 62117$ cm^{-1}, (PD) $= 21140$ cm^{-1}, (DS) $= 40977$ cm^{-1}.

The quadrupole integral s_q.—This was estimated by interpolating between the values of $s_q^{-1/2}$ for the $2p^2$ and $2p^4$ configurations (**5**). This gave

$$s_q = \tfrac{2}{5}\int_0^\infty r^2 P^2(2p)\, dr = 0{\cdot}273.$$

As a check, the wave function of Froese and Hartree (**4**) was used, giving $s_q = 0{\cdot}273$; this value was adopted. As a further check, the value of s_q for Ne V estimated (**5**) from other ions in the $2p^2$ series is $0{\cdot}235$; Froese and Hartree gave a wave function for Ne V and using it we obtain $s_q = 0{\cdot}240$. The excellent

agreement leads us to think that our estimate of s_q for Ne IV is as satisfactory as can be obtained at the present time.

Doublet intervals.—These were computed using formulae (16) of Aller, Ufford and Van Vleck (**2**). The results are listed in Table I, together with those of Horie(**7**) and the most recent experimental values (**8**, Appendix to Volume III). The agreement we obtain is quite good, and is rather better than that of Horie. In order to examine whether the results are sensitive to the adopted value of η, the calculations were repeated using $\eta = 4{\cdot}80$ cm^{-1}, all the other parameters remaining unaltered. The doublet intervals are given in Table I.

TABLE I

Doublet intervals in ^{2}P and ^{2}D terms

	$\Delta(^2P)$ (cm^{-1})	$\Delta(^2D)$ (cm^{-1})
This paper ($\eta = 4{\cdot}40$)	$+3{\cdot}0$	$-53{\cdot}1$
Horie (**7**)	-3	-65
This paper ($\eta = 4{\cdot}80$)	$-0{\cdot}8$	$-61{\cdot}5$
Experimental (**8**)	$+6{\cdot}4$	$-44{\cdot}8$

It is seen that the agreement between calculated and experimental values is much better for $\eta = 4{\cdot}40$ cm^{-1} than for $\eta = 4{\cdot}80$ cm^{-1}. Indeed, $\eta = 4{\cdot}00$ cm^{-1} would give nearly perfect agreement for the doublet intervals. However, because such a value would be well below that obtained by interpolation and from wave functions, we prefer to use $\eta = 4{\cdot}40$ cm^{-1} in further calculations.

Transition probabilities.—The transformation coefficients expressing the perturbed wave functions in terms of the unperturbed wave functions were obtained from equations (18) of (**2**) supplemented (**12**) by $a'' = 1$, $b'' = \zeta/(PS)$ and $c'' = (\sqrt{5})\zeta/2(PD)$. The transition probabilities were then obtained from formulae given by Shortley, Aller, Baker and Menzel (**13**, equations (2), (3), (11) and (12)). Our final results are given in Table II. Comparison with the few

TABLE II

Transition probabilities for [Ne IV]

Upper	Lower	A_m	A_q	A_{total}
$^2P_{3/2}$	$^2P_{1/2}$	$2{\cdot}3 \times 10^{-9}$	$2{\cdot}6 \times 10^{-22}$	$2{\cdot}3 \times 10^{-9}$
$^2P_{3/2}$	$^2D_{5/2}$	$0{\cdot}21$	$0{\cdot}19$	$0{\cdot}40$
$^2P_{3/2}$	$^2D_{3/2}$	$0{\cdot}36$	$7{\cdot}9 \times 10^{-2}$	$0{\cdot}44$
$^2P_{1/2}$	$^2D_{5/2}$	—	$0{\cdot}11$	$0{\cdot}11$
$^2P_{1/2}$	$^2D_{3/2}$	$0{\cdot}23$	$0{\cdot}16$	$0{\cdot}39$
$^2P_{3/2}$	$^4S_{3/2}$	$1{\cdot}33$	$1{\cdot}5 \times 10^{-7}$	$1{\cdot}33$
$^2P_{1/2}$	$^4S_{3/2}$	$0{\cdot}53$	$8{\cdot}6 \times 10^{-6}$	$0{\cdot}53$
$^2D_{3/2}$	$^2D_{5/2}$	$1{\cdot}4 \times 10^{-6}$	$1{\cdot}1 \times 10^{-17}$	$1{\cdot}4 \times 10^{-6}$
$^2D_{5/2}$	$^4S_{3/2}$	$1{\cdot}8 \times 10^{-4}$	$4{\cdot}1 \times 10^{-4}$	$5{\cdot}9 \times 10^{-4}$
$^2D_{3/2}$	$^4S_{3/2}$	$5{\cdot}3 \times 10^{-3}$	$2{\cdot}7 \times 10^{-4}$	$5{\cdot}6 \times 10^{-3}$

transition probabilities published by Pasternack (**11**) shows differences mainly due to revised doublet intervals and our lower value of s_q. Comparison with Naqvi's results (**9**) shows differences which may be ascribed to our use of empirical term separations; reasons have been given (**6**) which lead the writer to the view that it is better to use empirical term separations instead of attempting to determine the Slater integral F_2, as was done by Naqvi (cf. Naqvi and Talwar (**10**)).

The calculations were examined to see whether any of the transition probabilities were sensitive to the adopted values of the parameters. The quadrupole

parts of the $^2P_{3/2}$–$^4S_{3/2}$ and $^2P_{1/2}$–$^4S_{3/2}$ transitions appear to be somewhat sensitive to the value of η, but as the magnetic dipole components predominate for these two lines, the sensitivity has no practical consequence. None of the other transitions depends critically on the adopted parameters.

Intensities in the ^2P–^2D *multiplet.*—The relative intensities of the components of this multiplet have been estimated by Aller, Bowen and Minkowski (1) in the spectrum of NGC 7027. Using the transition probabilities given in Table II

Table III

Observed and calculated intensities in $^2P-^2D$ *multiplet*

Wavelength	Transition	Observed	Calculated
4714·26	$^2P_{3/2}-^2D_{5/2}$	0·8	1·1
4715·65	$^2P_{1/2}-^2D_{5/2}$	0·5	0·3
4724·16	$^2P_{3/2}-^2D_{3/2}$	1·3	1·3
4725·60	$^2P_{1/2}-^2D_{3/2}$	1·3	1·2

and his own estimates of the collision strengths, Dr M. J. Seaton has calculated the expected intensities of the lines of the ^2P–^2D multiplet. His results are listed in Table III; the theoretical intensities are normalized to the same total as the observed intensities. The agreement is well within the errors of calculation and observation.

Acknowledgment.—Thanks are due to Dr Seaton for suggesting this work, for contributing Table III, and for his continued interest in the writer's calculations.

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References

(1) L. H. Aller, I. S. Bowen, and R. Minkowski, *Ap. J.*, **122**, 62, 1955.
(2) L. H. Aller, C. W. Ufford, and J. H. Van Vleck, *Ap. J.*, **109**, 42, 1949.
(3) I. S. Bowen, *Ap. J.*, **121**, 306, 1955.
(4) Charlotte Froese and D. R. Hartree, *Proc. Camb. Phil. Soc.*, **53**, 663, 1957.
(5) R. H. Garstang, *M.N.*, **111**, 115, 1951.
(6) R. H. Garstang, *Proc. Camb. Phil. Soc.*, **52**, 107, 1956.
(7) H. Horie, *Prog. Theor. Phys.*, **10**, 298, 1953.
(8) Charlotte E. Moore, *Atomic Energy Levels*, National Bureau of Standards Circular 467.
(9) A. M. Naqvi, *Thesis*, Harvard, 1951, quoted by L. H. Aller, *Gaseous Nebulae*, Chapman and Hall, London, 1956, p. 188.
(10) A. M. Naqvi and S. P. Talwar, *M.N.*, **117**, 463, 1957.
(11) S. Pasternack, *Ap. J.*, **92**, 129, 1940.
(12) M. J. Seaton and D. E. Osterbrock, *Ap. J.*, **125**, 66, 1957.
(13) G. H. Shortley, L. H. Aller, J. G. Baker and D. H. Menzel, *Ap. J.*, **93**, 178, 1941.

CAN SPHERICAL CLUSTERS ROTATE?

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Summary

It is shown that contrary to former belief an unrelaxed spherical cluster of mass points may rotate without becoming oblate. When used of an unrelaxed system the argument that spherical objects do not rotate is thus false. Some present estimates of the Sun's circular velocity round the galaxy rest on this argument applied to the system of globular clusters.

Introduction.—We are here discussing clusters of gravitating mass points. Thus applications may be considered at several levels.

(i) Clusters of galaxies.

(ii) E0 galaxies themselves.

(iii) The system of globular clusters centred on the galactic centre.

(iv) The globular clusters themselves.

The system of globular clusters shows no pronounced flattening at the galactic poles; is this good evidence that it does not rotate? If it is not, then the justification for the determination of the Sun's circular velocity from the Doppler motions of the globular clusters is destroyed. These observations give the Sun's velocity in the coordinates in which the system of globular clusters does not rotate, and we have no longer any justification for assuming that this is not a rotating coordinate system.

I wish to show that unrelaxed systems may be steady, spherical, and rotating. Perhaps there are cosmogonical reasons why a spherical system may not rotate, but such a conclusion cannot be reached on dynamical grounds alone. It is possible that some systems with net angular momenta evolve into spherical states in times short compared with their relaxation time, before eventually flattening to the relaxed "gaseous" forms discussed by Jeans (1, 2). The form of the frequency function for the general spherical cluster is found in this paper. In general these clusters rotate. This is in contradiction to Jeans' result, but is obtained by using his method correctly and following the consequences.

My deepest thanks are due to Dr Woolley who not only suggested the problem†, but convinced me of the result. The argument that follows arose from stimulating and amusing discussions with him.

Consider a mighty Maxwell demon whose job it is to reverse the directions of motion of stars so that they describe their orbits in the reverse sense. We shall assume him capable of violating the law of angular momentum conservation.

* Received in original form 1959 May 6.

† The problem came originally from Dr Sandage.

On his entering a non-rotating spherical star cluster let him pick a direction and a sense of rotation about it. Let him reverse the velocities of all those stars which do not have the picked sense of motion about this axis. It seems likely that he can do this without disturbing the formerly spherical density distribution. If this is the case he succeeds in making a spherical but rotating cluster. The assumption implicitly made is that any non-rotating spherical cluster may be considered as two inter-penetrating clusters, rotating in opposite senses, *each of which is itself spherical.* (It is clear that they cannot be flattened since then the complete cluster would also be flattened.)

Jeans' theorem (1, 2, 3, 4).—This states that for a steady system of mass points moving independently in a potential V, the frequency function* f(which gives the density of representative points in the 6-dimensional phase space) must be a function of the independent first integrals of the equations of motion only. When V is spherically symmetrical 4 such integrals, the energy and the angular momenta, are readily obtained. A fifth may also be obtained but it may be shown that continuity of the distribution function forbids dependence on this integral†. We shall therefore neglect it.

We write for the position, velocity, velocity moment, and energy:

$\mathbf{r}=(x,y,z)$ Cartesian $=[r,\theta,\phi]$ spherical polar.

$\mathbf{c}=(u,v,w)$ Cartesian $=\{c,\theta_c,\phi_c\}$. θ_c,ϕ_c give the direction of $\mathbf{c}$.

$\boldsymbol{\varpi}=\mathbf{r}\times\mathbf{c}=(\varpi_1,\varpi_2,\varpi_3)=[\varpi,\theta_\varpi,\phi_\varpi]$. $\theta_\varpi,\phi_\varpi$ give the direction of $\boldsymbol{\varpi}$.

$\epsilon=c^2/2-V$.

We also use the notation $\int\ldots d^3c$ for $\iiint\ldots du\,dv\,dw$ and $\int\ldots d^3r$ similarly.

Jeans' theorem reads

$$f=f(\boldsymbol{\varpi},\epsilon). \tag{1}$$

We assume that stars of different masses are distributed with similar frequency functions. This gives the same result as assuming all stars to have the same mass $\bar{m}$.

The potential V may be made up of two parts;

(1) V_{self} the potential due to the cluster of moving mass points; and

(2) V_{ext} corresponding to an external field applied to the system.

Poisson's equation reads:

$$\nabla^2V_{self}=-4\pi\gamma\rho_{self}=-4\pi\gamma\bar{m}\int f\,d^3c. \tag{2}$$

Spherical clusters

Jeans argues that a spherical cluster is invariant under rotations. Hence f must be invariant. The independent invariants of $\boldsymbol{\varpi}$ and ϵ are ϖ^2 and ϵ and hence‡

$$f=f(\varpi^2,\epsilon). \tag{3}$$

* The cluster is completely described by its distribution in phase space. The density of this distribution is given by the frequency function which is often called the distribution function.

† A paper is in preparation on the justification for the neglect of the further integrals, such as this, in the use of Jeans' theorem (3).

‡ By the theorem that an invariant function is a function of the invariants. Our result may be obtained elementarily by partial differentiation (5).

However the condition that a cluster be spherical is not that f should be rotationally invariant but that ρ should be. The problem of finding the most general frequency function for spherical clusters thus entails finding the most general function $f(\varpi, \epsilon)$ for which $\frac{\rho}{\bar{m}} = \int f\, d^3c$ is a function of r alone.

There is one further condition; f must be nowhere negative. We shall call a cluster of form (3) a Jeans cluster. If also f is nowhere negative we say it gives a "real" Jeans cluster and if conversely an "imaginary" Jeans cluster.

Our first problem is that of fitting the density of a Jeans cluster to that of a given spherical cluster.

Theorem.—To any given spherical cluster (with frequency f_1) there corresponds a Jeans cluster (f_2) of the same density.

Proof.—Define

$$f_2 = \bar{f}_1 = \frac{1}{4\pi}\int\int f_1(\varpi, \epsilon) \sin\theta_\varpi \, d\theta_\varpi \, d\phi_\varpi.$$

Clearly f_2 is a function of (ϖ^2, ϵ) only and it thus gives a Jeans cluster. It is also non-negative (since f_1 was) and therefore gives a real Jeans cluster. It is obtained from f_1 by averaging over the directions $(\theta_\varpi, \phi_\varpi)$ of the normals to the planes of the stellar orbits. This averaging process merely tilts the orbital planes so that these normal directions are evenly distributed. This process does not change the mass or the speed distribution* in the spherical shell r to $r+dr$. Thus the radial variation of density is the same in f_1 and f_2. Since they both give spherical clusters the density only varies radially and they give the same density everywhere.

We define the deviation Δf by:

$$\Delta f = f_1 - f_2.$$

By our theorem

$$\int \Delta f \, d^3c \equiv 0 \text{ all } \mathbf{r}. \tag{4}$$

The general Jeans cluster is known, thus the general spherical cluster may be found if we can find the most general form of the deviation Δf.

We wish therefore to find the most general Δf satisfying (4) and subject to the restriction

$$f_2 + \Delta f \geqslant 0. \tag{5}$$

This last may be considered more as a restriction on the type of Jeans cluster (f_2) which may be added to a given deviation than a restriction on the deviation itself. We therefore consider deviations satisfying (4).

A particular class of solutions.—Consider a Δf antisymmetric in $\boldsymbol{\varpi}$, i.e.

$$\Delta f(\boldsymbol{\varpi}, \epsilon) = -\Delta f(-\boldsymbol{\varpi}, \epsilon)$$

$\boldsymbol{\varpi} = \mathbf{r} \times \mathbf{c}$, ϵ is even in $\mathbf{c}$, thus for any given $\mathbf{r}$ this Δf is antisymmetric in $\mathbf{c}$. Hence (4) is satisfied. If we add an $f_2(\varpi^2, \epsilon)$ such that (5) is satisfied we will have succeeded in making a non-Jeans spherical cluster. Note that creation of such a Δf antisymmetric in $\boldsymbol{\varpi}$ is just what is achieved by the demon on reversing stars in their orbits. In general he has not been given freedom to turn all the

* Speed = | velocity |. The result will be needed later.

stars one way, but is no doubt chewing his claws with discontent over not being allowed to finish the job. Later we shall find clusters which cannot be obtained by letting a demon act on a Jeans cluster.

We take as a model the Jeans cluster due to Schuster (**6**).

(1) Frequency function

$$f_2 = A\left(V - \frac{c^2}{2}\right)^{7/2}$$

where

$$A = \frac{48}{7\sqrt{2}}\frac{a^2}{\pi^3}\frac{M}{\bar{m}}\left(\frac{1}{\gamma M}\right)^5.$$

(2) Total mass M, mean radius $\bar{r} = 2a$.

(3) Potential

$$V = \frac{\gamma M}{\sqrt{a^2 + r^2}}.$$

Density

$$\rho = \frac{3a^2 M}{4\pi(a^2 + r^2)^{5/2}}.$$

(4) R.m.s. stellar velocity

$$\sqrt{\overline{c^2}} = \frac{\sqrt{3}}{4}\pi\left(\frac{\gamma M}{2a}\right)^{1/2}.$$

(5) Average magnitude of the angular momentum of a star taken about any axis

$$\bar{m}\bar{\varpi} \simeq \bar{m}\frac{\sqrt{2}}{7}(2a)\sqrt{\frac{\gamma M}{2a}}.$$

Total angular momentum of cluster $H = 0$.

Consider the deviation given by

$$\Delta f = \frac{\varpi_3/h}{1 + (\varpi_3/h)^2} f_2$$

where $h > 0$ is a constant. Since the minimum of $\dfrac{x}{1 + x^2}$ is $-\frac{1}{2}$ hence $f_2 + \Delta f \geqslant 0$. Thus, since Δf is antisymmetric in $\boldsymbol{\varpi}$, $f_1 = f_2 + \Delta f$ is a cluster with the same density as f_2 everywhere. Its angular momentum is

$$H = \iint \varpi_3 f_1 \, d^3c \, d^3r = \iint \Delta f \varpi_3 \, d^3c \, d^3r = h \iint \frac{(\varpi_3/h)^2}{1 + (\varpi_3/h)^2} f_2 \, d^3c \, d^3r > 0.$$

We have succeeded in making a rotating cluster. A different cluster for which the integrals are tractable is the demon's cluster. This however has a discontinuous deviation

$$\Delta f = Sgn(\varpi_3) f_2 \quad \text{where} \quad Sgn(\varpi_3) \begin{array}{ll} = +1, & \varpi_3 > 0, \\ = -1, & \varpi_3 < 0. \end{array}$$

The properties (1) ... (4) hold for this cluster too but the total angular momentum

$$H = \bar{m} \iint |\varpi_3| f_2 \, d^3c \, d^3r$$

since $\varpi_3 \; Sgn \, \varpi_3 = |\varpi_3|$. On integration

$$H = \frac{2048}{49 \cdot 99 \cdot \sqrt{2}} \bar{m}\left(\frac{M}{\bar{m}}\right) a \left(\frac{\gamma M}{a}\right)^{1/2} \simeq \frac{\sqrt{2}}{7} \bar{m}\left(\frac{M}{\bar{m}}\right)(2a)\left(\frac{\gamma M}{2a}\right)^{1/2}$$

On comparison with (5) we see that the contributions from the individual stars all add, rather than subtract, as the demon requires.

The general spherical star cluster.—We shall need the following mathematical theorem.

Theorem.—The most general function defined on a sphere, whose integral around every great circle vanishes, is any function antisymmetric on inversion in the sphere's centre, i.e. if the sphere is $|\hat{\mathbf{r}}|=1$

$$g(\hat{\mathbf{r}})=-g(-\hat{\mathbf{r}}).$$

To prove this theorem we need the concept of the great circle transform which I will now develop.

To each pair of antipodes $\mathbf{r}$, $-\mathbf{r}$ on the sphere there corresponds one great circle, their equator. Thus the space whose elements are great circles is similar to the space of antipodal points. To make up for this pairing of the points we shall until further notice only deal with functions on the sphere which are symmetric under inversion. The great circle transform of a function $f(\theta,\phi)$ is defined by

$$\check{f}(\alpha,\beta)=\frac{1}{2\pi}\int_{C_{\alpha\beta}} f(\theta,\phi)\,ds_{\alpha\beta}$$

where $ds_{\alpha\beta}$ is the infinitesimal path length round the great circle $C_{\alpha\beta}$ which has (α,β) as one of its poles. Since the choice of pole is arbitrary $\check{f}$ is symmetric. Thus we have a transformation on the sphere which, to any symmetric function, yields a corresponding symmetric function. We shall here assume that there exists an inverse transformation* such that given $\check{f}$ we can find f.

Suppose f is the symmetric part of a function whose integral round every great circle vanishes. Then from the definition of the great circle transform $\check{f}=0$. However since there is an inverse transform this defines f. In particular since the transformation is linear we have $f=0$. Clearly the antisymmetric part of the solution, g say, can be anything since a great circle through P will always pass through P's antipodes. Hence the theorem as stated.

We now use our theorem to prove the result for star clusters. We left this problem at equation (4) where we reduced it to the problem. What is the most general function $\Delta f(\boldsymbol{\varpi},\epsilon)$ satisfying

$$\int \Delta f\, d^3c=0.$$

Measure θ_c, ϕ_c, from $\hat{\mathbf{r}}$ as pole $\mathbf{c}=(c,\theta_c,\phi_c)$ we require

$$\int\!\!\int\!\!\int \Delta f\left(rc\sin\theta_c, \hat{\boldsymbol{\varpi}}, \frac{c^2}{2}-V(r)\right)\sin\theta_c\, d\theta_c c^2\, dc\, d\phi_c \equiv 0$$

where

$$\hat{\boldsymbol{\varpi}}=\frac{\boldsymbol{\varpi}}{|\boldsymbol{\varpi}|}=\widehat{\mathbf{r}\times\mathbf{c}};$$

note $\hat{\boldsymbol{\varpi}}$ is independent of $|r|$, $|c|$, θ_c. It merely gives the direction of the normal to the $\mathbf{r}$, $\mathbf{c}$ plane (in 3 space). We may thus perform the first two integrations, defining

$$g(r,\hat{\boldsymbol{\varpi}})\equiv \int\!\!\int \Delta f\left(rc\sin\theta_c, \hat{\boldsymbol{\varpi}}, \frac{c^2}{2}-V\right)\sin\theta_c\, d\theta_c\, c^2\, dc$$

and obtain

$$\int g(r,\hat{\boldsymbol{\varpi}})\, d\phi_c=0 \quad \text{all } \mathbf{r}. \tag{6}$$

* I have a rigorous proof of this result but it is neither short nor beautiful.

Now $\hat{\boldsymbol{\varpi}}$ is the normal to the plane $\phi_c = \text{const}$ whereas all such planes pass through $\hat{\mathbf{r}}$; thus this last integration is in effect over all directions $\hat{\boldsymbol{\varpi}}$ perpendicular to a given $\hat{\mathbf{r}}$. Since r is not involved in the integration, regard it as a fixed parameter. Then translating (6) into words:—

A function of direction is such that when integrated over all directions perpendicular to a given $\hat{\mathbf{r}}$ the result vanishes. However the given $\hat{\mathbf{r}}$ is arbitrary; thus $g(r, \hat{\boldsymbol{\varpi}})$ is a function of direction such that its integral over all directions $\perp^{ar}$ to *any* given direction vanishes. Alternatively g is a function defined on a sphere, whose integral round any great circle vanishes. Hence by our theorem

$$g(r, \hat{\boldsymbol{\varpi}}) = -g(r, -\hat{\boldsymbol{\varpi}}) \tag{7}$$

i.e. the symmetric part of g vanishes.

Split Δf into symmetric and antisymmetric parts writing $\Delta f = \Delta_+ f + \Delta_- f$ where $2\Delta_+ f = \Delta f(\boldsymbol{\varpi}, \epsilon) + \Delta f(-\boldsymbol{\varpi}, \epsilon)$, (7) implies

$$\iint \Delta_+ f \sin\theta_c \, c^2 \, dc \equiv 0.$$

Since $\hat{\boldsymbol{\varpi}}$ is not involved in the integration we may now regard it as a fixed parameter. Write $\Delta_+ f = \Delta_+ f(\hat{\boldsymbol{\varpi}}; \varpi^2, \epsilon)$. (7) declares that, for each $\hat{\boldsymbol{\varpi}}$, $\Delta_+ f$ is the difference between two Jeans distributions of the same density. There are many Jeans distributions $(f(\varpi^2, \epsilon))$ corresponding to a given density function*

$$\rho(r) = \bar{m} \int f \, d^3c = 2\pi\bar{m} \iint f \sin\theta_c \, d\theta_c c^2 \, dc.$$

Any distribution of the form $f_2 + \Delta_+ f$ may thus be obtained by having different corresponding Jeans distributions for each coplanar set of motions ($\hat{\boldsymbol{\varpi}}$.) Finally we obtain the general distribution by adding any antisymmetric $\Delta_- f$ provided the total $f_1 = f_2 + \Delta_+ f + \Delta_- f$ is nowhere negative. Summarizing our method we have the following steps.

(1) Choose a density function $\rho(r)$; Poissons equation (2) yields V.

(2) Find the Jeans distributions corresponding to that density*.

(3) Consider a function $f_3(\hat{\boldsymbol{\varpi}}, \varpi^2, \epsilon)$ which for each $\hat{\boldsymbol{\varpi}}$ reduces to one of the above set of Jeans distributions. f_3 may be found by assigning one of this set to each direction $\hat{\boldsymbol{\varpi}}$. f_3 gives the general spherical cluster distribution symmetric in $\boldsymbol{\varpi}$.

(4) $f_1 = f_3 + \Delta_- f$ is the general spherical cluster distribution, where $\Delta_- f$ is antisymmetric in $\boldsymbol{\varpi}$ and is restricted to make $f_3 + \Delta_- f \geqslant 0$ everywhere.

In short, take the Jeans distributions of the right density; assign one such to each orbital plane. Let the demon reverse some motions, and we have a way of constructing the general spherical cluster.

The fastest rotating clusters

"Given the density $\rho(r)$ of a cluster what is the greatest angular momentum it can hold consistent with its maintaining that density?"

"What is the spherical cluster of greatest angular momentum irrespective of density distribution?"

This latter is rather a wonderful figment of the imagination: a great spherical shell of stars round which each star crawls with a circular velocity which is quite minute. However all their senses of rotation are the same about the demon's

* See Appendix.

axis, and though their speeds are small they carry large angular momenta. We now return to the saner question posed above. We first demonstrate the existence of spherical clusters in which all the stars move in circles. By considering the orbits in each plane we then show that circular orbits can carry more angular momentum about a given axis that any other distribution of orbits that give rise to the same density. This is sufficient to demonstrate our result.

We shall assume the given density distribution to be such that the circular orbits are everywhere stable.

Since $\rho(r)$ is given, hence $V(r)$ is known, hence $r(\epsilon)$ (the radius corresponding to circular orbits of energy ϵ) is known; and $\varpi[\epsilon]$ (the angular momentum in circular orbit of energy ϵ) is known.

It is a well-known property of the stable circular orbit that "Of all the orbits of given energy the circular orbit carries the greatest angular momentum ϖ".

Construct a Jeans cluster as follows:

$$f(\varpi^2, \epsilon) = 0 \quad \text{unless} \quad \varpi^2 = (\varpi[\epsilon])^2.$$

This ensures that all the stars move in circular orbits. By giving $f((\varpi[\epsilon])^2, \epsilon)$ as a function of ϵ we may populate each spherical shell just as we wish. In particular we do this so that we obtain the required density

$$\rho(r). \tag{8}$$

We have thus proved that:

"To each cluster there corresponds a Jeans cluster with the same density everywhere and in which all the stars move in circles."

Consider the planar motion of a star. This motion gives rise to a certain distribution of density with radius "$\rho(r)$" and a certain angular momentum $\bar{m}h\cos\lambda$ about an axis at λ to the plane's normal. Can we replace this star by a distribution over orbits which taken together give the same density $\rho(r)$ but a greater angular momentum $\epsilon\varpi\cos\lambda$? In particular does replacement with circular orbits have the desired effect?

$\delta t = \dfrac{\delta r}{\dot{r}}$ gives the time spent by the star in $r \to r + dr$. Let r_{min} and r_{max} be the limits of this star's orbit. Thus

$$\frac{\dfrac{dr}{\dot{r}}}{\displaystyle\int_{r_{min}}^{r_{max}} \frac{dr}{\dot{r}}}$$

gives the "number of stars" which, permanently within $r \to r + dr$, will give the same average density there. Their angular momenta is given by

$$\frac{\bar{m}\varpi(r)\dfrac{dr}{\dot{r}}\cos\lambda}{\displaystyle\int \frac{dr}{\dot{r}}}$$

where $\bar{m}\varpi(r)$ is the circular angular momentum at distance r. The total angular momentum gained by such replacements is

$$\bar{m}\left\{\frac{\displaystyle\int_{r_{min}}^{r_{max}} \varpi(r)\frac{dr}{\dot{r}}\cos\lambda}{\displaystyle\int_{r_{min}}^{max} \frac{dr}{\dot{r}}} - h\cos\lambda\right\}.$$

The replaced system thus carries more angular momentum than the original provided (calling the integral J)

$$J\cos\lambda \equiv \int_{r_{min}}^{r_{max}} (\varpi(r)-h)\frac{dr}{\dot{r}}\cos\lambda > 0,$$

but

$$\frac{\dot{r}^2}{2}+\frac{h^2}{2r^2}-V=E$$

and

$$\frac{(\varpi(r))^2}{r^3} = -\frac{\partial V}{\partial r}$$

thus the integral J is

$$\int_{r_{min}}^{r_{max}} (\sqrt{r^3(-V')}-h)\frac{dr}{\sqrt{2(E+V)-h^2/r^2}},$$

$$J=\int_{r_{min}}^{r_{max}} \frac{(-V'-h^2/r^3)\,dr}{r^{-3/2}(\sqrt{-V'}+h/r^{3/2})\sqrt{2(E+V)-h^2/r^2}}.$$

Integrating by parts ($\sqrt{2(E+V)-h^2/r^2}$ vanishes at end points)

$$J=0+\int_{r_{min}}^{r_{max}} \sqrt{2(E+V)-h^2/r^2}\,\frac{d}{dr}[(\sqrt{-V'}+h/r^3h)^{-1}r^{3/2}]\,dr,$$

therefore

$$J=-\int_{r_{min}}^{r_{max}} \frac{\sqrt{2(E+V)-h^2/r^2}}{\left(\sqrt{\frac{-V'}{r^3}}+h/r^3\right)^2}\left[\frac{d}{dr}\left(\sqrt{\frac{-V'}{r^3}}\right)-\frac{3h}{r^4}\right]dr$$

but

$$\frac{d}{dr}\left(-\frac{V'}{r^3}\right)=\frac{d}{dr}(-V'r^3)\frac{1}{r^6}+\frac{6V'}{r^4}<0,$$

since

$$\frac{d}{dr}(-V'r^3)<0$$

(circular orbit stability) and $V'<0$ always. Thus

$$\left[\frac{d}{dr}\left(\sqrt{\frac{-V'}{r^3}}\right)-\frac{3h}{r^4}\right]<0.$$

hence $J>0$. We may take $\lambda \leqslant \pi/2$ by choosing the sense in which the stars describe their orbits. Thus $J\cos\lambda>0$. We have therefore shown that angular momentum is always gained in the replacement of non-circular orbits by circular orbits giving the same density, i.e. the orbits carrying greatest angular momentum for given density are circular and the net angular momentum is greatest if the senses in which they are described are arranged by the demon. Thus:

"The fastest rotating spherical cluster of given density is the 'circular demon cluster'."

It is interesting to note that if all the stars travelled at the same speeds, at the same central distances, and all contributed their angular momentum in the same direction instead of at all angles up to $\pi/2^c$, then the total angular momentum would be twice that of the above cluster. This indicates how little is the relief granted to a rotating cluster when it flattens.

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Appendix

How to find all the Jeans distributions of a given density:

Find one such distribution, e.g. construct the Jeans cluster in which all the stars move in circles as in (8)*. Let its distribution of speeds with radius be $F_2(c,r)$, i.e.

$$\bar{m}4\pi \int F_2(c,r)c^2\,dc = \rho(r).$$

Take an arbitrary function $\Delta' F(q,r)$ satisfying

$$\int \Delta' F(q,r)\,dq \equiv 0 \quad \text{all } r$$

write $q=c^3$ and $\Delta' F(c^3,r)=\Delta F(c,r)$ then

$$\int \Delta F c^2\,dc \equiv 0.$$

$F_1=F_2+\Delta F$ has the same density as F_2 and all the F with this property may be so obtained. We show that to each such F_1 there corresponds a Jeans distribution. Given $F_1(c,r)$ we wish to construct a Jeans distribution $f_2(\varpi^2,\epsilon)$ such that

$$\iint f_2 \sin\theta_c\,d\theta_c\,d\phi_c = 4\pi F_1,$$

i.e.

$$\int_0^{\pi/2} f_2((rc\sin\theta_c)^2,\epsilon)\sin\theta_c\,d\theta_c = F_1(c,r). \tag{9}$$

Method.—Since ρ is known we may solve Poisson's equation for V_{self} and add V_{ext} (if any) to obtain $V(r)$ as a known function. Now $c^2/2-V(r)=\epsilon$ and we write $c^2r^2=\chi$, $\chi/2r^2-V(r)=\epsilon$ solving for r we obtain $r=r(\chi,\epsilon)$, evidently

$$c^2=\frac{\chi}{[r(\chi,\epsilon)]^2};$$

note c being a magnitude $\geqslant 0$. Write

$$F_1\{c,r\}=F_1\left\{\frac{\chi^{1/2}}{r(\chi,\epsilon)},\ r(\chi,\epsilon)\right\}=F(\chi,\epsilon).$$

Write $\chi\sin^2\theta_c=T$ then (9) becomes

$$\int_0^{\chi}\frac{f_2(T,\epsilon)\,dT}{2\chi\sqrt{1-T/\chi}}=F(\chi,\epsilon)$$

or

$$\int_0^{\chi}\frac{f_2(T,\epsilon)\,dT}{\sqrt{\chi-T}}=2\sqrt{\chi}F(\chi,\epsilon).$$

* Or construct (following Kurth, *Zs. f. Astrophys.*, **26**, 168, 1949) the distribution function, $f(\epsilon)$ dependent on energy only, which corresponds to the given density.

This is Abel's integral equation and has solution

$$f_2(T,\epsilon)=\frac{1}{\pi}\int_0^T \frac{\partial/\partial\chi(2\sqrt{\chi}F)\,d\chi}{\sqrt{T-\chi}}$$

where the differentiation is performed at constant ϵ. Thus we have found the Jeans distribution as required. We may remark however that most such distributions will be imaginary since we have no proof that f_2 is nowhere negative. We will have to throw away all these imaginary solutions and leave only the real ones before passing to the next step in obtaining the general distribution.

Clare College,
Cambridge:
1959 *June* 21

References

(1) Jeans, J. H., *Problems of Cosmogony and Stellar Dynamics*, Cambridge, 1919.
(2) Jeans, J. H., *Astronomy and Cosmogony*, Cambridge, 1928.
(3) Chandrasekhar, S., *Principles of Stellar Dynamics*, Chicago, 1942.
(4) Smart, W. M., *Stellar Dynamics*, Cambridge, 1938.
(5) Kurth, R., *An Introduction to the Mechanics of Stellar Systems*, Pergamon, 1957.
(6) Schuster, Brit. Ass. Report, 1883, p. 428. Schuster was here dealing with gaseous rather than stellar spheres.

PHOTOMETRY IN THE MAGELLANIC CLOUDS, II. SOME COLOUR-MAGNITUDE ARRAYS

R. v. d. R. Woolley

(Communicated by the Astronomer Royal)

(Received 1959 September 16)

Summary

Colour-magnitude arrays are given for two small areas, and for some clusters, in the Large Magellanic Cloud. They are the result of measurement of photographic plates secured with the Radcliffe 74-inch reflector by Sandage and with the Victoria telescope of the Royal Observatory at the Cape by Alexander, and are referred to photoelectric sequences determined by Eggen and Sandage. The arrays show remarkable numbers of very blue stars.

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1. Standard sequences in two areas in the Large Magellanic Cloud were determined by Eggen and Sandage and were described in Paper I of this series*. A number of photographic plates were exposed by Sandage with the 74-inch reflector at the Radcliffe Observatory, Pretoria, and by Alexander with the Victoria telescope at the Royal Observatory at the Cape of Good Hope. The latter is a twin telescope consisting of a photographic refractor of aperture 24 inches and a photovisual refractor of aperture 18 inches, both having a focal length of approximately 23 feet. The photoelectric sequences are used to calibrate the magnitudes measured on these plates with the Sartorius Iris Diaphragm photometer of the Royal Greenwich Observatory. Generally speaking, the Pretoria plates are used to construct colour-magnitude arrays and the Cape plates to investigate variable stars, but some overlap of purpose has been necessary. The present note discusses four colour-magnitude arrays, as follows.

(i) A field of stars, including most of the photoelectric sequence LMC Field No. 1 of Paper I and measuring about $16' \times 14'$ centred on R.A. $5^h\ 00^m$, Dec. $-65°\ 34'$ (1950), was examined. The magnitudes of 327 stars were measured on a pair of Pretoria plates, the telescope having been diaphragmed to 43 inches. The exposures were 15 minutes both on the visual plate (103aD + GG11 filter) and the blue plate (103aO + GG13 filter). The magnitudes found in this way, added to those of 21 stars in the photoelectric sequence, are plotted in Fig. 1, which shows $B-V$ plotted against V.

(ii) A similar field, of area $12' \times 11'$ centred approximately on R.A. $5^h\ 32^m$, Dec. $-67°\ 08'$ (1950), was chosen for examination. It includes some of the stars in LMC Field II of Paper I. Magnitudes were formed from measures on three Cape visual plates (103aD + Omag 301) and three Cape blue plates (103aO + GG13). The results are plotted in Fig. 2.

* Eggen and Sandage, *M.N.*, **120**, 79, 1960.

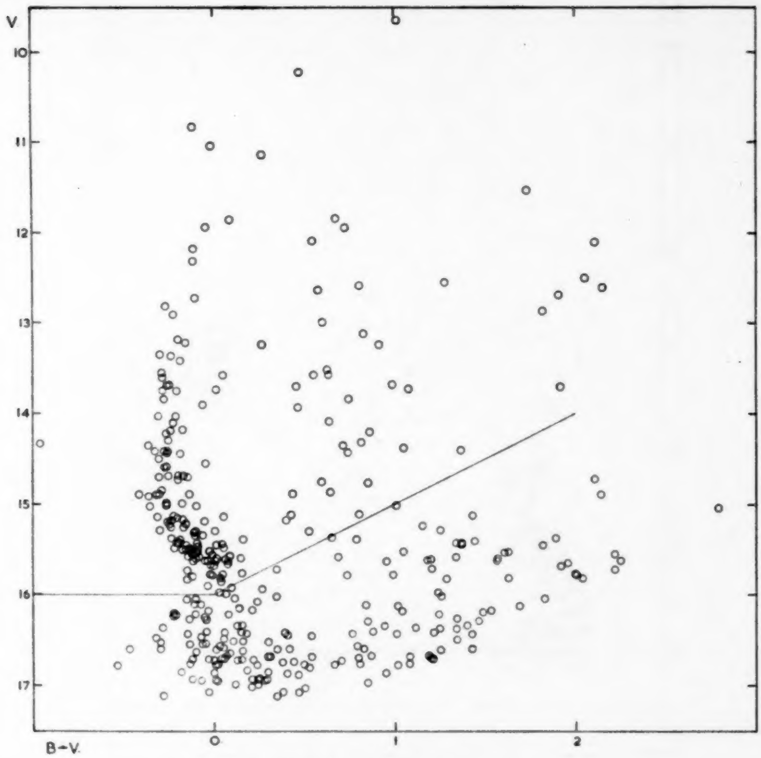


FIG. 1.—*Colour-magnitude array, Field I.*

(iii) The cluster NGC 2004 (R.A. 5^h 29^m, Dec. $-67°$ 19′ (1950)) was examined. This cluster is described in NGC as " globular, bright, pretty large, pretty rich, compressed, and having stars of twelfth magnitude ". It is illustrated in Plate 6 (*a*). This cluster is on the Cape plates of Field II and on four available Radcliffe plates. Sixteen stars marked in the illustration were chosen to form a sub-sequence and their magnitudes adopted by comparison with some stars of the Field II p.e. sequence occurring on each of two *B* and two *V* Radcliffe plates (the telescope being stopped to 43 inches). A total of 91 stars were measured on three pairs of Cape plates and the two pairs of Radcliffe plates, relative to the sub-sequence as standards. The adopted magnitudes of the sub-sequence stars are shown in Table I. Of the stars measured, one proved to be an eclipsing variable. The resulting colour-magnitude array is shown in Fig. 3.

(iv) Lastly, the neighbouring globular clusters NGC 1818 and NGC 1810 were examined. NGC 1818 is described as " globular, very bright, pretty large, rich, very much compressed and partially resolved ". It is situated at R.A. 5^h 04^m, Dec. $-66°$ 30′ (1950), and is illustrated in Plate 6 (*b*). The nearby cluster NGC 1810 is " considerably faint, small, round, and little brighter in the middle ".

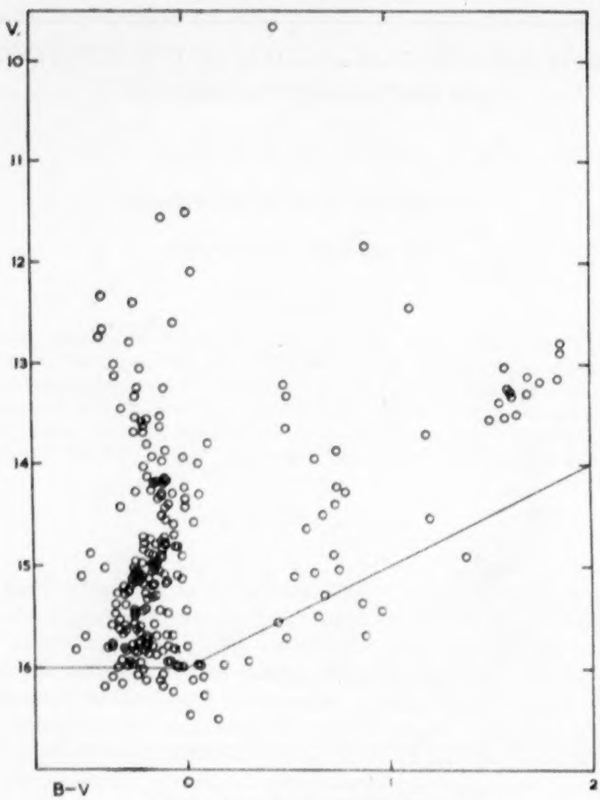


FIG. 2.—*Colour-magnitude array, Field II.*

TABLE I

Sequence for NGC 2004: provisional values

Star	*B*	*V*	*B−V*
	m	m	m
a	11·70	12·06	−0·36
b	15·14	15·48	−0·32
c	13·38	13·66	−0·38
d	14·67	14·92	−0·25
e	11·75	12·10	−0·35
f	13·50	13·68	−0·18
g	15·10	15·34	−0·24
h	14·24	14·34	−0·10
i	14·88	15·17	−0·29
j	14·79	15·06	−0·27
k	14·56	14·86	−0·30
l	15·17	15·39	−0·22
m	15·36	15·68	−0·28
n	15·72	15·70	+0·02
o	15·56	15·62	−0·06
p	15·55	15·74	−0·19
q	16·58	—	—

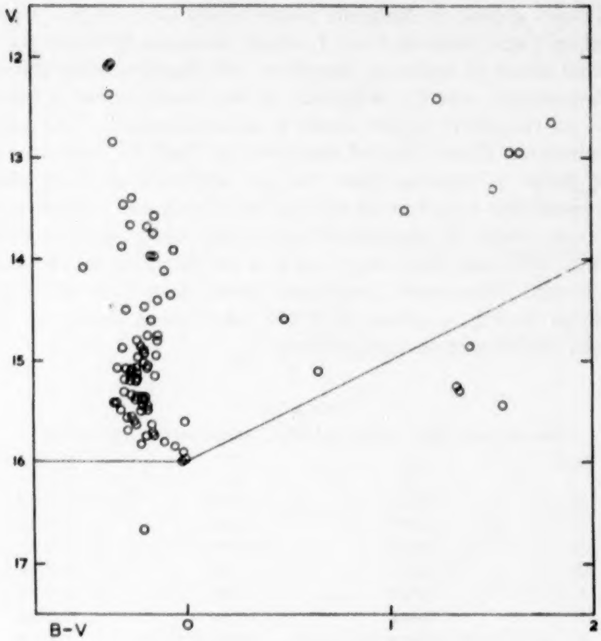


FIG. 3.—*Colour-magnitude array, NGC 2004.*

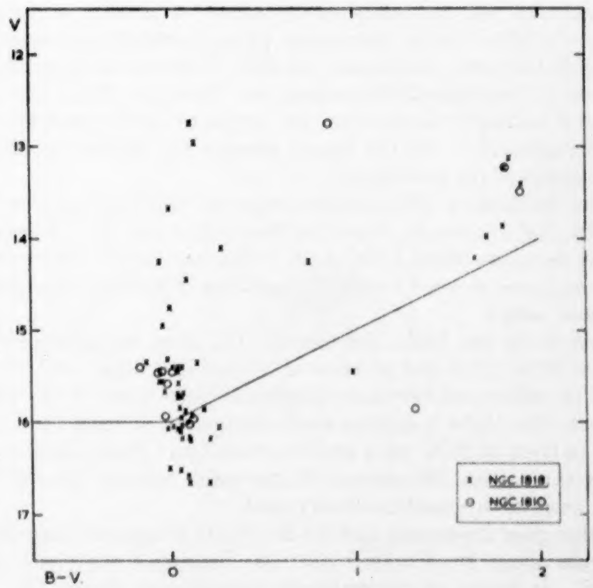


FIG. 4.—*Colour-magnitude arrays, NGC 1818 and NGC 1810.*

These clusters appear on Radcliffe plates which have no p.e. sequence stars on them, and on Cape plates of Field I, which of course have the p.e. sequence, which is distant about 3½ inches on the plate—the clusters being about 1½ inches from the plate centre, and the sequence, in the mean, about 2 inches. (The exposed area on the plates is just under 6 inches square.) The sub-sequence shown reproduced in Plate 6 (*b*) and described in Table II was formed by taking the mean of about 10 transfers from the p.e. sequence on Cape plates. It is intended to check this sequence by setting up a fresh p.e. sequence near NGC 1818 which may result in systematic corrections being applied to the results found for NGC 1818 and NGC 1810, but it is not thought that these corrections will exceed $0^{m}\cdot10$. The mean magnitudes found from four pairs of Radcliffe plates are shown in Fig. 4, those for NGC 1818 being shown as crosses and those for NGC 1810 being shown as circles.

TABLE II

Sequence for NGC 1818 *and NGC* 1810: *provisional values*

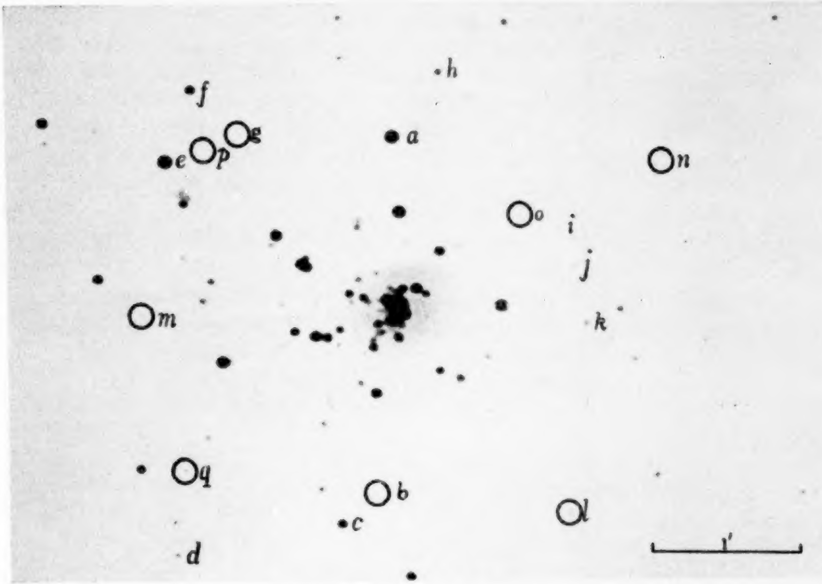
Star	*B*	*V*	*B*−*V*
	m	m	m
a	13·35	13·28	+0·07
b	16·16	16·06	+0·10
c	15·76	15·69	+0·07
d	15·50	15·36	+0·14
e	14·92	14·96	−0·04
f	15·44	15·40	+0·04
g	16·04	15·86	+0·18
h	16·32	16·05	+0·27
i	16·57	16·52	+0·05

The magnitudes will be published *in extenso* on another occasion. In all cases, the results below visual magnitude 16 and photographic magnitude 16½ must be regarded as rather uncertain—because of nearness to the plate limits and paucity of stars in the photoelectric sequences. However, Fig. 1 depends on two Radcliffe plates measured close to the p.e. sequence, and is probably reliable to another half magnitude. All the figures show a line indicating a (rather conservative) estimate of the plate limit.

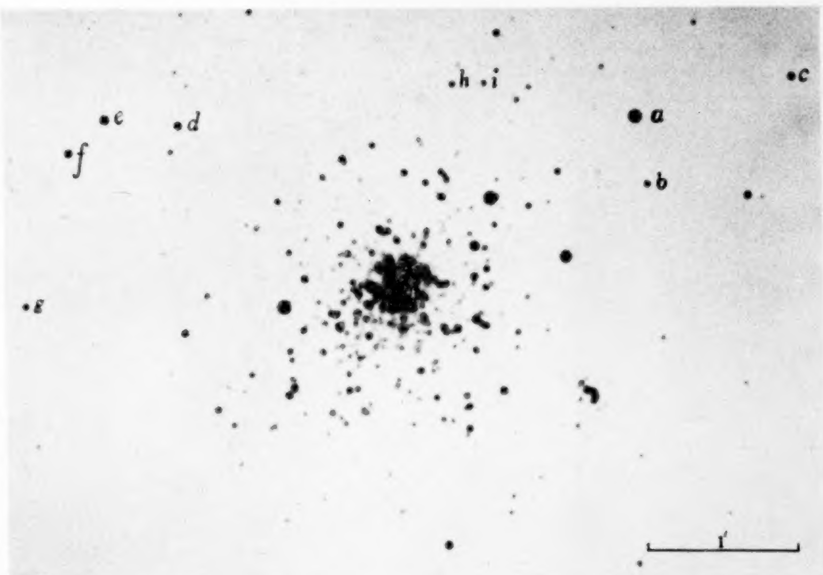
2. All four fields show a remarkable sequence of blue giant stars and a few red giant stars, and a few stars of intermediate colour which are probably not for the most part members of the LMC at all. The numbers of these stars of intermediate colour agree generally with the numbers of galactic stars per unit area given in Seares' tables.

The arrays of the two fields, and that of NGC 2004, are practically identical. The arrays of NGC 1818 and of 1810 are indistinguishable, and are displaced about $0^{m}\cdot25$ in colour and $0^{m}\cdot60$ in magnitude from those of the other fields, which suggests that there is slightly more absorption in front of these clusters than there is in front of NGC 2004 and the two fields. From the blueness of the stars in these three cases, the amount of absorption between us and these parts of the LMC must be supposed to be very small.

3. The blue giant sequences, and the red giants at apparent magnitudes 13 to 14 and colours about $B-V=1^{m}\cdot8$ resemble Sandage's schematic array for *h* and χ Persei. In terms of contemporary evolutionary theory, the areas and objects concerned are extremely young.



(a) NGC 2004 : provisional magnitude sequence



(b) NGC 1818: provisional magnitude sequence

R. v. d. R. Woolley, Photometry in the Magellanic Clouds: II

4. Some interest attaches to the question whether NGC 1818 and NGC 2004 are globular. Since most, if not all, galactic globular clusters show red-giant luminosity arrays, as does for example M67, and are classed as " very old " or " extreme population I " objects, some may choose to classify clusters by their colour-luminosity arrays. To the present writer, however, it seems necessary to classify clusters (in principle) by referring to their internal motions, and to regard a cluster as globular if the great majority of the stars in it have not got the velocity of escape from the gravitational field of the cluster—and as open if the cluster is a moving association of stars, most of which have the velocity of escape from the others. If one accepts this dynamical concept of the difference between globular clusters and open clusters, and accepts the evolutionary view that clusters like M67 are red because they are old, one is led to the view that all globular clusters must have been blue once. Perhaps NGC 1818 and NGC 2004 are simply new globular clusters. They have roughly the same diameter as NGC 1783, which is a " typical " globular cluster without blue giant stars—namely, about one minute of arc, or about 18 parsecs—and the total brightness of NGC 1818 appears (by inspection of the *B* plates) greater than that of NGC 1783. The latter may, however, contain the greater number of faint stars.

It would be of great interest to determine the distribution of brightness of NGC 1818 with radius, to see whether it could be reconciled with the hypothesis that the stars in NGC 1818 are distributed subject to their mutual gravitation.

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1959 *August* 31.

THE SYNTHESIS OF LARGE RADIO TELESCOPES

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Summary

Many investigations in radio astronomy are limited by the resolving power which can be achieved by conventional methods of aerial construction.

A new method of obtaining increased resolving power has been developed, which has been applied to the construction of both " pencil-beam " systems and interferometers. In this method two aerials are arranged so that their relative position may be altered to occupy successively the whole area of a much larger equivalent aerial. By combining mathematically the information derived from these different positions, it is possible to obtain a resolving power equal to that of the large equivalent aerial. Since the combination of the individual records may be done with different phase relationships, it is possible, without extra observations, to " scan " the synthesized aerial over an appreciable solid angle; because of this the total observing time of a synthesized instrument is of the same order as that of a conventional instrument.

An interferometric system designed for the study of radio stars has been built which has an equivalent area for resolution of $8 \cdot 10^5$ sq. ft as well as a "pencil-beam" system with an equivalent area of $3 \cdot 10^6$ sq. ft. The sensitivity of both systems corresponds to a " collecting area " of about $2 \cdot 10^5$ sq. ft.

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1. *Introduction.*—The operation of a radio telescope is limited by two dominating factors. One is the sensitivity, which specifies the smallest radio flux which may be detected, and the other is the angular resolving power. The relative importance of these factors is strongly dependent upon the type of investigation undertaken and for this reason specialized instruments are often developed for particular applications. In studies of the solar radio emission, for example, where high resolving power is needed at metre wave-lengths the primary requirement is that of angular resolution; the use of conventional aerials such as paraboloids is clearly impracticable. Even if structures large enough to give the required resolution were mechanically feasible their energy-collecting area would be far in excess of that needed to detect the relatively intense solar radio emission. Instruments have therefore been developed in which high angular resolution has been obtained by dispersing small aerials over a wide area of ground. Such systems include the diffraction grating type of array (Christiansen 1953; Blum, Boischot and Ginat 1958) and the interferometer of variable spacing (Stanier 1950, Machin 1951, O'Brien 1953); these two methods achieve the same result but in the former case all the aerials are present simultaneously, while in the latter case two aerials only are moved, successively, to the different positions of the grating.

Similar techniques have been developed for the study of the distribution of the galactic emission. Conventional systems have been successfully employed for the short wave-lengths, but in many problems, particularly those involving the distribution of radio "brightness" at high galactic latitudes, it is important to make observations at wave-lengths of several metres; by dispersing the available collecting area it is again possible to obtain much larger resolving powers with adequate sensitivity. Two systems have been used, one of which employs a pair of mutually perpendicular thin apertures (Mills and Little 1953); the other utilizes a single thin aperture in conjunction with a small movable aerial (Blythe 1957). An earlier version of the latter instrument was used by Scheuer and Ryle (1953) to obtain increased resolving power in one coordinate for studying the distribution of radio emission across the galactic plane.

The observation of radio stars presents a more complex problem in which three main approaches have been made. (*a*) The use of conventional systems of moderate size at a wave-length sufficiently short to provide the required angular resolution (Piddington and Trent 1956; Seeger, Westerhout and Conway 1957; Roman and Yaplee 1958). These observations have been characterized by very small signal/noise ratios, and in consequence long integration times are used and extensive surveys have not been possible. (*b*) The use of conventional aerials of larger collecting area at longer wave-lengths (Hanbury Brown and Hazard 1953; Ryle and Hewish 1955); the signal/noise ratio of such systems has usually been large and the observations have been limited by resolution. (*c*) The use of "unfilled" systems such as the Mills Cross (Mills and Little 1953) at a long wave-length. The angular resolution of such a system may be very large but the collecting area is smaller.

If any major increase in the depth of radio star surveys is to be made it is necessary to construct systems with a collecting area exceeding 50 000 ft^2 capable of operating at shorter wave-lengths or to use, at long wave-lengths, "unfilled" systems having larger collecting areas than the Mills Cross.

Since the flux density of the sources falls with wave-length and since the difficulties of constructing an aerial of given physical size get very much greater if it is to operate at short wave-lengths, it is possible that even with the new receiver techniques now being developed (Adler 1958, Giordmaine *et al.* 1958) it may be difficult to attain a sufficient improvement by using conventional systems at short wave-lengths. The alternative approach, which utilizes a long wave-length and correspondingly large collecting areas, appears to offer greater possibilities.

In this paper an account is given of a new principle in the design of large radio telescopes. The method, which has already been used successfully by Blythe (1957), makes use of two aerials arranged successively in different configurations to provide information equivalent to that obtained from the use of an aerial of much greater physical size. Apart from a considerable economy of structure this method avoids some of the difficulties associated with the physical achievement of a graded excitation of amplitude and phase which is required in the case of large extended arrays such as the Mills Cross. Besides allowing greater collecting areas to be realized, the shape of the reception pattern can be adjusted, by computation alone, to suit different types of observation. The method necessarily involves considerable computation but this does not present a serious problem with the large electronic computers now available.

2. *The principle of aperture synthesis.*—It is well known that a simple interferometer, consisting of a pair of non-directional receiving elements connected to a receiver, gives a response proportional to one Fourier component of the two-dimensional distribution of radio brightness across the sky. By taking measurements in which both the spacing and the orientation of the interferometer are varied it is possible to determine the two-dimensional transform of the brightness distribution within certain limits and hence, by Fourier inversion, to derive the brightness distribution as observed with a specified resolving power (O'Brien 1953). Aperture synthesis may be regarded as a logical extension of this method but it will here be discussed in a somewhat different, but essentially equivalent, way which may give a more direct understanding of the method.

Consider, for example, the operation of a receiving aperture such as a large two-dimensional array, or paraboloidal reflector. The signal which is delivered to a receiver from such an aerial may be regarded as the vector addition of the currents induced in each portion of the aperture. In the case of a paraboloid this vector addition is achieved, automatically, at the focal point, while for an array the same result is achieved by connecting all the elements to the receiver in the same phase. When the direction of an incident wave does not coincide with the normal to the aperture plane the induced current in the elements of the aperture will suffer a progressive phase shift and vector addition then gives rise to the usual directional properties of the aperture. In the case of an array, it it possible to add the currents with their phases suitably adjusted, and in this way to vary the direction of the principal response of the aperture; this feature is of great importance in large aerial systems where physical tilting of the aperture plane may be impossible.

Now if it were possible to measure, separately, the current induced at each portion of the aperture by moving a small aerial successively across it, vector addition of the currents would give, for a constant source, exactly the same result as that obtained by using the complete large aperture. This process is not possible using a single receiving element since measurement of the phase of the induced current raises difficulties. By using two small aerials arranged as an interferometer, however, the relative phases are readily obtained and the successive determination of the current in this way forms the basis of the synthesis method.

(*a*) *The synthesis of a single aperture.*—Consider a single rectangular aperture such as that shown in Fig. 1. The resultant current induced in the aperture by an incident plane wave is $\sum I_n e^{i\phi_n}$ where ϕ_n is the phase of the wave at the position of the nth element. The power P delivered to a receiver is then given by

$$P \propto \sum I_n e^{i\phi_n} \sum I_n e^{-i\phi_n} \propto \sum I_n^2 + \sum I_m I_n \cos(\phi_m - \phi_n). \tag{1}$$

If all the elements are of the same size, the first term in expression (1) is simply N times the power derived from a single element; the terms of type $I_m I_n \cos(\phi_m - \phi_n)$ are just the output obtained when a phase-switching receiver is connected to the elements m and n through equal cables (Ryle 1952). A single measurement of the power induced in one element, suitably combined with the outputs from a phase-switching receiver connected to a pair of elements, arranged successively to cover all possible combinations of positions, then gives a result

exactly equivalent to using the whole large aperture. By performing the addition with the value of $(\phi_m - \phi_n)$ adjusted to give a progressive phase shift across the aperture it is possible to vary the direction of the maximum response of the aperture in a manner analogous to tilting the aperture plane. It is thus possible, without the necessity for further observations, to scan the synthesized aperture through an angle which is only limited by the directional properties of the small element. To perform the addition with modified values of $(\phi_m - \phi_n)$ it is also necessary to measure terms of the type $I_m I_n \sin(\phi_m - \phi_n)$, and these are obtained by connecting the pair of elements in phase quadrature to the phase-switching receiver.

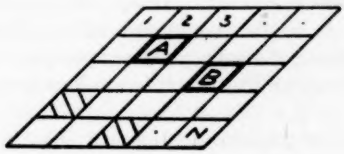


FIG. 1.—*The synthesis of a rectangular aperture.*

The measurements might proceed in the following way: having determined the power induced in a single element, an interferometer is set up with one element A fixed at position 1 and the sine and cosine terms are obtained by placing the other element B successively in all the remaining positions. Element A is then moved to position 2, etc., and the same procedure carried out. Now if this were done it is immediately obvious that many arrangements of A and B, such as those shaded in Fig. 1 are identical. A procedure for obtaining the different terms with no repetition is indicated in Fig. 2 (*a*) where the element A is fixed and element B is arranged to cover, once only, all the positions contained in a rectangle of approximately twice the area. It is of course possible to move both elements, in which case all arrangements can be obtained without moving the elements outside a rectangle of side D. It may be noted that if a total of N observations was required initially, the number now needed with no repetition is $(2N)^{1/2}$.

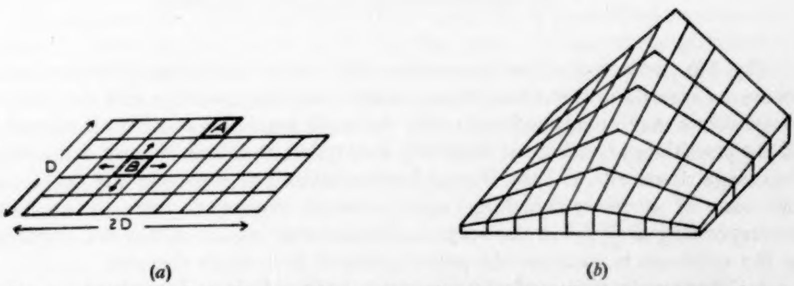


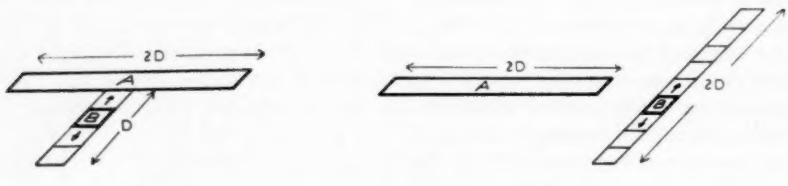
FIG. 2.—(*a*) *One method of synthesizing a rectangular aperture without repeating any element configuration.*

(*b*) *The weighting function for each position of the movable element.*

To obtain results equivalent to using the complete aperture the different terms must now be added with a weighting factor appropriate to the number of repetitions of each configuration of the elements. The form of the weighting function appropriate to each position of the movable element is sketched in Fig. 2 (*b*). Direct addition of the terms without such a tapered weighting function gives a result equivalent to the use of an "optimum" array as discussed by Arsac (1955) and Barber (1958). While such systems give the greatest ratio of information to noise they are sometimes undesirable in practical measurements owing to the large subsidiary maxima inherent in their reception patterns.

When the elements used in the synthesis have a width much greater than one wave-length, addition of the different terms with a progressive phase-shift is equivalent to the use of an aperture having a discontinuous "stepped" phase distribution across it. As in the case of an echelon diffraction grating this gives rise to more than one principal maximum in the reception pattern. This difficulty can be avoided if it is arranged that the different locations of the movable element overlap slightly.

It is not necessary for the elements A and B to have the same shape and there are sometimes practical reasons why it is preferable for one of them to extend over the complete width of the required aperture so that synthesis is only performed in one dimension. A system of this type, which is exactly equivalent to that shown in Fig. 2 (*a*) is shown in Fig. 3 (*a*). When this method is adopted it is necessary to arrange that the current induced in each element of the extended array has the value appropriate to the weighting function shown in Fig. 2 (*b*). The aperture must consequently have a tapered distribution of excitation along its length.



(*a*) (*b*)

FIG. 3.—(*a*) *One-dimensional synthesis of a single aperture.*
(*b*) *One-dimensional synthesis of an interferometer.*

(*b*) *The synthesis of an interferometer.*—For many investigations it is preferable to use an interferometer rather than a single receiving aperture and the method of synthesis may be applied in exactly the same way. By an obvious extension of the preceding arguments it is readily shown that an interferometer comprising two large apertures of side D may be synthesized as shown in Fig. 3 (*b*). In the case of a phase-switching interferometer, however, there is no term corresponding to $\sum I_n^2$ in the output of the receiver and thus it is not necessary in the synthesis to measure the power received by a single element.

3. *Observation time and signal-to-noise ratio.*—Before describing practical applications of the synthesis method it is necessary to consider how it compares with conventional methods in regard to the observation time and signal to noise

ratio. Since a synthesis using small elements requires many different measurements it might be expected that the observing time is necessarily longer. The time required for the repeated observations is, however, almost exactly compensated by the increased scanning rate which may be employed. It is useful to define a quantity which gives a direct comparison of the performance of synthesized and conventional instruments having the same angular resolution. In principle, the observation of a given area of sky can, by suitably arranging the scanning rate and the receiver time constant, be made to occupy a certain interval of time. We can therefore define the "efficiency" of a synthesized instrument as the ratio of its absolute sensitivity (i.e. signal/noise ratio) as compared with that of a conventional instrument of the same resolving power when the same observing time is allowed in each case. The efficiencies of various types of synthesized instrument will now be discussed.

(*a*) *Scanning rate and observation time.*—During a survey a radio telescope is generally scanned continuously on one coordinate, either by steering the instrument or by using the Earth's rotation, and a given region of the sky is investigated strip by strip. For our purpose it is more convenient to imagine that the scanning is performed in discrete intervals in both coordinates, the instrument being directed towards a certain point in the sky for the required integration time and them moved instantaneously to the next point. The angular distance between adjacent points in the scanning lattice depends upon the resolving power of the instrument and Bracewell (1956) has shown that the greatest interval allowed when the aerial is a rectangular aperture of sides a and b is given by $\lambda/2a$ and $\lambda/2b$ in the respective coordinates, where λ is the wave-length. No additional information is gained by using a finer scanning lattice but information is lost if a coarser lattice is adopted.

Suppose, now, that a given solid angle Ω of the sky is scanned using a square aperture of side D. The total observing time is given by $4\Omega D^2\tau/\lambda^2$ where τ is the integration time of the receiver. If the sky is scanned using the equivalent synthesized instrument in which the small elements are square apertures of side d, the time required to scan the solid angle for each of the arrangements is $4\Omega d^2\tau/\lambda^2$ where it is assumed that the receiver time constant is the same as before. The number of different arrangements is approximately $2D^2/d^2$ and the total observing time for the same receiver time constant is therefore $8\Omega D^2\tau/\lambda^2$. This is seen to be only twice that required had the complete aperture been used and it is important to notice that it is independent of the size of the small elements.

To calculate the observing time when synthesis is performed in only one dimension, consider the system shown in Fig. 4 (*a*). If one of the elements is a rectangle of sides d_1 and d_2, and the other an extended aperture of width d_1 and length $2D$, then the scanning intervals are $\lambda/2d_1$ and $\lambda/2D$ *. The total number of arrangements is approximately D/d_1 and the observing time is therefore $4\Omega D^2\tau/\lambda^2$. The observing time is thus half that needed when both elements were small and the same as if a complete aperture had been used. It is also of interest to consider the system shown in Fig. 4 (*b*) which is similar to an aerial constructed by Mills except that one arm of the cross is omitted. Although no

* The extended aperture of length $2D$, when used with the appropriate tapered excitation is equivalent, when scanning, to a uniform aperture of length D. More exactly, the scanning interval is $\lambda/2D+d_2$ but we assume $D \gg d_2$.

synthesis is required such a system yields the same (non-repetitive) information as do the synthesized apertures under discussion and it will be shown that it has approximately the same efficiency. We note here that the scanning interval is $\lambda/2D$, as for the complete aperture, so that the observation time is again $4\Omega D^2\tau/\lambda^2$.

It should be noted that this analysis assumes that the bandwidth of the receiver is not too great and that the area of sky under investigation is large enough to contain more than one coarse scanning interval $\lambda/2d$. The effect of a finite bandwidth is to narrow the reception pattern of the individual elements at large spacings (Ryle and Vonberg 1948), and this may necessitate the use of a finer scanning lattice. In the case of a single aperture it may be shown that the effect is negligible provided that $\Delta f/f$ is appreciably less than d/D; for an interferometer the ratio $\Delta f/f$ must be less than the ratio of d to the maximum element spacing.

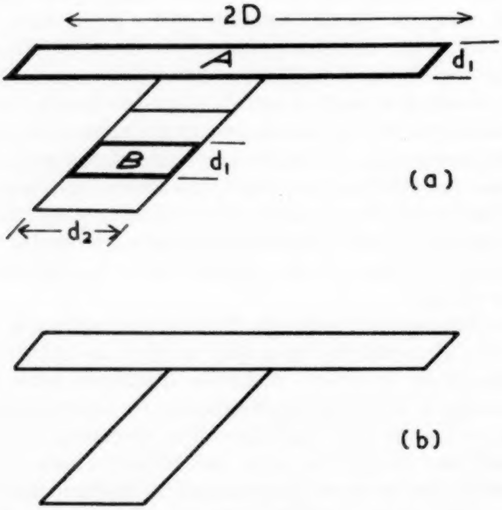


FIG. 4.—(*a*) *The synthesis of a rectangular aperture.* (*b*) *A similar system which may be compared to the Mills Cross. It can be regarded as a synthesis in which all configurations are present simultaneously.*

(*b*) *The efficiency of synthesized systems.*—To derive the efficiency of a synthesized aperture we note that the signal obtained with a complete aperture of side D is directly proportional to its area D^2. If, for an integration time τ, the root mean square receiver noise is n then the signal/noise ratio is proportional to D^2/n.

When the aperture is synthesized using small elements of area d^2 the signal obtained by adding the $2D^2/d^2$ different observations with appropriate weights is proportional to D^2. Since the receiver noise is unrelated for the different observations the resultant noise is $2^{1/2}nD/3d$ where weighted addition has been performed. The signal/noise ratio is thus given by $3Dd/2^{1/2}n$. Since however, the observing time for a fixed receiver time constant was twice that required when using a conventional aperture, the efficiency is given by $3d/2D$.

In the case where a one-dimensional synthesis is used, as in Fig. 4 (*a*), the signal obtained for each arrangement of A and B is proportional to $2(Dd_1^2d_2)^{1/2}$ since the equivalent area is given by the geometric mean of the two apertures (Ryle 1952). The number of arrangements is D/d_1 and the signal/noise ratio is $D(3d_1d_2)^{1/2}/n$. In this case the observing time was the same as with a conventional aperture and so the efficiency is $(3d_1d_2)^{1/2}/D$. For the system of Fig. 4 (*b*), in which no synthesis is employed, the signal is $D(2d_1d_2)^{1/2}$ and so the signal/noise ratio is $D(2d_1d_2)^{1/2}/n$ and hence the efficiency is $(2d_1d_2)^{1/2}D$. All the systems shown in Figs. 3 and 4 thus have a comparable efficiency and the maximum possible efficiency in a synthesized aperture is $(d/D)^{1/2}$ which is attained when $d_2=D$.

While the efficiency of a synthesized aperture is necessarily less than that of a conventional aperture of the same resolving power this should not be regarded as an immediate disadvantage. As mentioned earlier, the efficiency of a conventional aperture is often far higher than is required for a particular investigation so that the same physical aperture, arranged as a synthesized system of adequate efficiency and increased angular resolution, would have provided a much more powerful instrument.

4. *The practical application of the method*

(*a*) *General considerations.*—Since systems derived by one- or two-dimensional synthesis have approximately the same performance there might seem to be considerable advantages with a two-dimensional system arising from the economy of physical structure. There are, however, a number of practical disadvantages with two-dimensional synthesis and these are discussed below.

(i) Data handling and computation: for a one-dimensional synthesis involving N arrangements of the elements, the corresponding two-dimensional system would require N^2 arrangements. The computational and data-handling problems are thus enhanced.

(ii) Scanning: the angular scanning rate of a two-dimensional system must be considerably higher than for a one-dimensional system of the same resolving power for a survey to occupy a given observing time (cf. Section 3). It is a great practical convenience to employ the Earth's rotation for scanning in right ascension, but this fixed scanning rate may be too slow for the desired observational programme. Facility for pointing the elements in both coordinates is then necessary which raises practical difficulties which may be avoided in a one-dimensional system.

(iii) Receiver linearity: in a two-dimensional synthesis each intense source will be received over a solid angle λ^2/d^2, and it may be difficult to ensure a sufficiently linear overall response to allow observation of weak sources within this area. In the case of one-dimensional synthesis the area of sky affected in this way is reduced by the receptivity of the long aerial to λ^2/Dd.

(iv) Bandwidth: a limitation particularly relevant to the synthesis of an interferometer is set by the condition $d/D>\delta f/F$ (cf. Section 3). Since the overall spacing of the synthesized apertures will be several times larger than their individual aperture D, the maximum bandwidth permitted in a two-dimensional synthesis is necessarily smaller than for the corresponding one-dimensional system in which the long aerial runs East–West.

(*b*) *The design of systems utilizing the Earth's rotation.*—For the reasons outlined above systems adopting one-dimensional synthesis have a number of practical advantages. In current applications long thin apertures running

East–West, with provision for rotation about the long axis, may be constructed relatively economically and so the remaining discussion will be restricted to the design of such systems; it is possible, however, that other problems might reach a simpler solution if two-dimensional methods were employed.

At first sight the restriction of the design to what is essentially a transit instrument, in which the scanning rate is fixed by the Earth's rotation, might be thought to lead either to inadequate sensitivity due to a too short integration time, or to an unnecessarily long observing time. While it is true that the observing time required to survey a given area of sky is determined entirely by the angular resolution of the system and the rotation speed of the Earth, it is, however, possible to adjust the efficiency so that this scanning rate is always the optimum for a particular application. It is assumed that the area of sky involved includes 24^h in right ascension and an angle of at least λ/d in declination.

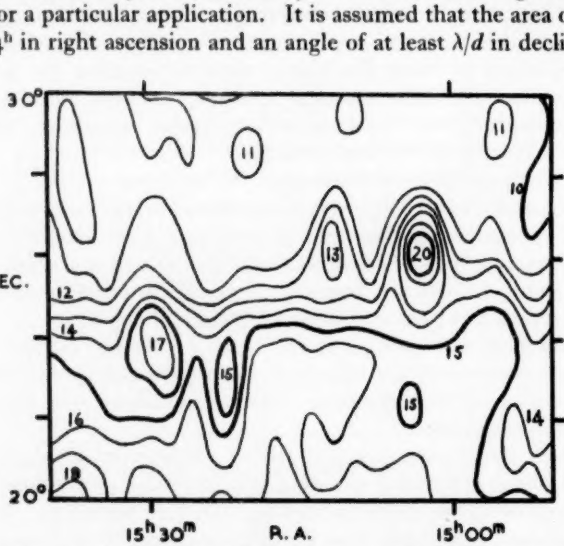


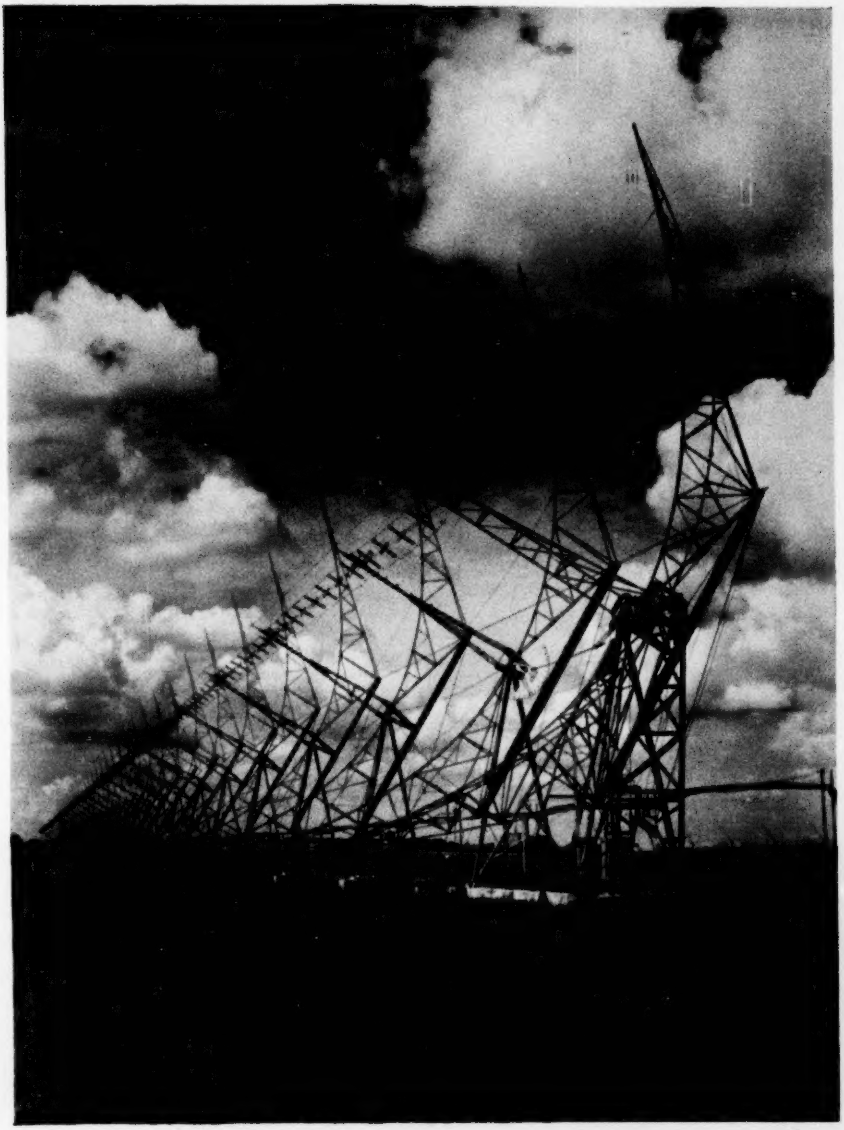
Fig. 5.—*Part of a map obtained with the 7·9 m "pencil-beam" system. The contours are in units of 1000 °K.*

In Section 3 the efficiency of a one-dimensional system such as that in Fig. 3 was shown to be proportional to the quantity $(d_1d_2)^{1/2}/D$. Once the efficiency has been determined, the relative values of d_1 and d_2 may be decided by considerations of structural economy provided that their product is constant.

5. *The design of two practical systems*

(*a*) *A pencil beam aerial for a wave-length of* 7·9 *m*.—This instrument was designed to provide a high resolution survey of the general galactic background radiation at a long wave-length so that, in conjunction with other observations at shorter wave-lengths, reliable information would be available on the spectral distribution of the emission from many galactic features.

The original instrument built by Blythe (1957) employed one-dimensional synthesis as in Fig. 4 (*a*) with an East–West array of length $D = 1200$ ft and a movable element of length d_1 = one wave-length. The effective width d_2 was less than the wave-length so that a single series of observations provided a complete map of the accessible sky. However, the presence of ground reflections led to an uncertainty in the receptivity in declination and gave an uncertainty in the overall scaling of brightness at southerly declinations.



The long fixed element of the radio star interferometer.

In the new instrument, which was designed to give an angular resolution of $0^\circ{\cdot}8 \times 0^\circ{\cdot}8$, a fixed aerial of length 3300 ft and width 40 ft was used so that a well-defined receptivity in declination was achieved at declinations above -10°. This increase in width and the use of a movable element of length 100 ft gave an efficiency of 7 per cent.

A preliminary survey of limited resolving power ($0^\circ{\cdot}8 \times 1^\circ{\cdot}6$), in which spacings of the movable aerial up to 30λ were used, has been carried out and part of the map is shown in Fig. 5.

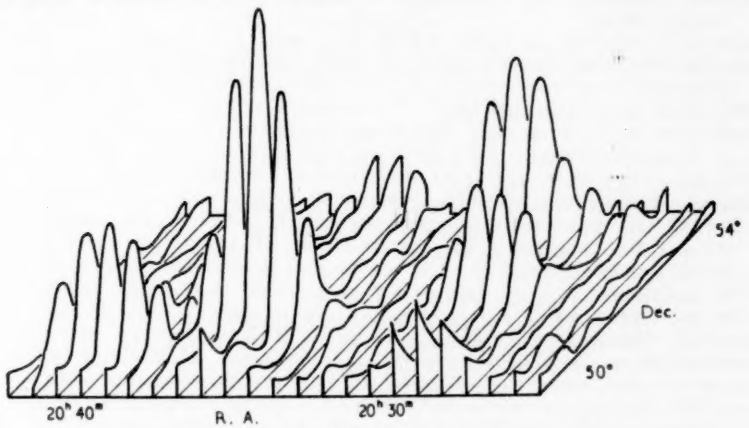


FIG. 6.—*A section of the first map obtained with the radio star interferometer. The most intense source at* $20^h\,37^m$ *has a flux density of* 11×10^{-26} *w.m.*$^{-2}$ *(c/s)*$^{-1}$.

(*b*) *A radio star interferometer on a wave-length of* 1·7 *m.*—A system capable of carrying out a deeper survey of radio sources than those made hitherto Mills and Slee 1957, Edge *et al.* 1959) was required, and considerable increases of both resolution and sensitivity were necessary. In addition the design was influenced by the importance of obtaining the positions of sources with greater accuracy, particularly in declination, and by the necessity for obtaining good measurements of the angular diameter of the most intense sources.

An interferometric system having an approximately symmetrical primary reception pattern was adopted, in which the resolution in right ascension was provided by a 1450 ft parabolic trough running East–West (as in Plate 7). The required efficiency (about 25 per cent) was decided from an extrapolation of the number versus flux density counts of previous surveys; the ratio of the width d_1 of both aerials, and the length d_2 of the movable aerial were governed (cf. Section 4) by consideration of the relative costs of the parabolic elements and of the railway track supporting the movable aerial (Ryle 1960).

A section of the first map obtained with this instrument is shown in Fig. 6. Some observations to establish accurate positions of 64 of the most intense radio stars have already been described (Elsmore, Ryle and Leslie 1960).

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Cavendish Laboratory,
Cambridge:
1959 *August.*

References

Adler, R., 1958, *Proc. I.R.E.*, **46**, 1300.
Arsac, J., 1955, *Opt. Acta.*, **2**, 112.
Barber, N. F., 1958, *New Z. J. Sci.*, **1**, 35.
Blum, E. J., Boischot, A., and Ginat, A., 1957, *Ann. d'Astrophys.*, **20**, 115.
Blythe, J. H., 1957, *M.N.*, **117**, 644.
Bracewell, R. N., 1956, *Aust. J. Phys.*, **9**., 297.
Brown, R. Hanbury, and Hazard, C., 1953, *M.N.*, **113**, 123.
Christiansen, W. N., 1953, *Nat re*, **171**, 831.
Edge, D.O., McAdam, W. B., Shakeshaft, J. R., Baldwin, J. E., and Archer, S., 1959, *Mem. R.A.S.*, **68**, 37.
Elsmore, B. E., Ryle, M. and Leslie, P. R. R., 1959, *Mem. R.A.S.*, **68**, 61.
Giordmaine, J. A., Alsop, L. E., Mayer, C. H., and Townes, C. H., 1958, *Proc. I.R.E.*, **47** 1062.
Machin, K. E., 1951, *Nature*, **167**, 889.
Mills, B. Y., and Little, A. G., 1953, *Aust. J. Phys.*, **6**, 272.
Mills, B. Y. and Slee, O. B., 1957, *Aust. J. Phys.*, **10**, 162.
O'Brien, P. A., 1953, *M.N.*, **113**, 597.
Piddington, J. H., and Trent, G. H., 1956, *Aust. J. Phys.*, **9**, 481.
Roman, N. G., and Yaplee, B. S., 1958, *Proc. I.R.E.*, **46**, 199.
Ryle, M., 1952, *Proc. Roy. Soc.* A, **211**, 351.
Ryle, M., 1960, *J.I.E.E.*, **6**, 14.
Ryle, M., and Hewish, A., 1955, *Mem. R.A.S.*, **67**, 97.
Ryle, M., and Vonberg, D. D., 1948, *Proc. Roy. Soc.* A, **193**, 98.
Seeger, C. L., Westerhout, G., and Conway, R. G., 1957, *Ap. J.*, **126**, 585.
Scheuer, P. A. G., and Ryle, M., 1953, *M.N.*, **113**, 3.
Stanier, H. M., 1950, *Nature*, **165**, 354.

THE ABSORPTION COEFFICIENT OF A PLASMA AT RADIO FREQUENCIES

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Summary

The theory of radiation due to free–free transitions by electrons in a plasma has been the subject of much work. Elwert has given an exact theory for binary electron-ion encounters but there has been doubt about the applicability of the formulae when an electron moves in the field of many ions at comparable distances; it is the aim of this paper to remove that doubt.

The power radiated by an electron at frequency ν is proportional to the mean square amplitude of the Fourier component of frequency ν of its acceleration; as Cohen, Spitzer and Routly have pointed out in a different context, one obtains the correct value of this mean square amplitude by treating all encounters as independent binary encounters, provided that the collision parameters concerned are all less than the Debye length. In the radiation problem, one is only interested in frequencies greater than the plasma frequency, and the binary collisions contributing to such radiation all have collision parameters less than the Debye length. Consequently Elwert's treatment always gives the correct result, except at frequencies just above the plasma frequency. An explicit formula for the absorption coefficient of a plasma is worked out for a Maxwellian distribution of electron speeds.

In the last section, the earlier considerations are applied to find the conditions necessary for radio-frequency radiation from encounters between electrons and concentrations of space charge.

1. *Introduction.*—The emission and absorption of radio waves by ionized hydrogen is of considerable importance in astronomy, both in connection with the radio emission from the solar corona and in connection with the emission (especially at centimetre and decimetre wave-lengths) and absorption (at wave-lengths of 10 m and more) of radiation by interstellar H II clouds. With the recent development of low-noise amplifiers, it may be expected that greater precision will be achieved in high-frequency observations than has hitherto been possible, and particularly for this reason it seems desirable to have an accurate theory of the emission and absorption of radio waves in a plasma.

Much has been written on the subject, but I shall not attempt to give the history of the problem, and refer only to the work of Elwert (1) and of Smerd and Westfold (2). Elwert's work deserves particular attention, because he derives an exact quantum mechanical formula for the radiation from collisions between electrons of initial velocity v and positive ions of charge Ze. He then shows that the exact formula contains Gaunt's quantum mechanical formula (3) as one limiting case, and a classical formula as another limiting case; he also shows that the classical formula is an excellent approximation to the exact formula in radio-astronomical applications. Elwert's result has, however, been open to one objection. Occasionally an electron moving through a plasma passes so

close to an ion that its motion is controlled predominantly by the field of that ion (and much of the radiation is emitted in these close encounters), but generally the electron moves in a field to which several ions make comparable contributions, so that a theory which considers only binary collisions is not applicable. Smerd and Westfold took the view that, since the fields of different ions will, in general, tend to cancel out, one must consider only encounters with collision parameters b less than the mean interionic distance ($b_1 = N_i^{-1/3}$). Collision parameters do not occur explicitly in Elwert's quantum mechanical calculation, and Smerd and Westfold adopt a different approach, which is purely classical. They also introduce the effect of the refractive index of the plasma for the first time. Smerd and Westfold's procedure is not convenient for the present discussion, so it is proposed to show in outline how the classical formula for the absorption coefficient may be derived from first principles, taking "multiple collisions" into account, and at the same time carrying out the calculation to a better approximation.

2. *Definitions.*—Various quantities which will occur in the calculation are defined below, in particular, various characteristic lengths (b_1 to b_4) of the plasma. v denotes the speed of an electron, $-e$ its charge, m its mass. N_i and N_e are the numbers of ions and electrons, respectively, in unit volume. Ze is the charge of an ion; k is Boltzmann's constant. T is the kinetic temperature of the electrons. b is the collision parameter, which is defined for a binary encounter as the distance of the ion from either asymptote of the electron's (hyperbolic) trajectory.

b_0 is a value of b chosen to be $\gg b_4$ and $\ll$ various upper limits (see below).

$b_1 = N_i^{-1/3}$, the mean interionic distance.

$b_2 = \dfrac{(\overline{v^2})^{1/2}}{2\pi\nu}$; the distance travelled by a typical electron in one period of the radiation (frequency ν) under consideration, is $2\pi b_2$.

$b_3 = \left(\dfrac{3kT}{4\pi n_e Z^2 e^2}\right)^{1/2} = \dfrac{(\overline{v^2})^{1/2}}{2\pi\nu_0}$ is the Debye shielding length; ν_0 in the second expression is the plasma frequency.

b_4 is given by $\dfrac{Ze^2}{b_4} = \dfrac{3}{2}kT = \frac{1}{2}m(\overline{v^2})$.

At distances $\lesssim b_4$ from an ion, an electron's potential energy is comparable with its kinetic energy. An encounter with collision parameter $\lesssim b_4$ scatters the electron through a large angle.

3. *Description of the physical processes.*—Suppose now that an electron is moving through a plasma; provided that the kinetic temperature of the ions is not much greater than that of the electrons, the electrons move much faster than the ions, and (to the accuracy to which we are working) the ions may be regarded as stationary.

Every time an electron passes close to an ion, it is deflected; neglecting radiation damping, it performs a hyperbolic orbit (Fig. 1 (*a*)). During this encounter, it suffers accelerations whose components vary with time as shown qualitatively in Fig. 1 (*b*) and (*c*). The frequency spectra of these acceleration

components are also illustrated; the "transverse" acceleration, Fig. 1 (*b*) and 1 (*d*), has a white spectrum up to frequencies of the order of v/b but cuts off at higher frequencies. The "longitudinal" acceleration is only important at frequencies of the same order of magnitude as v/b.

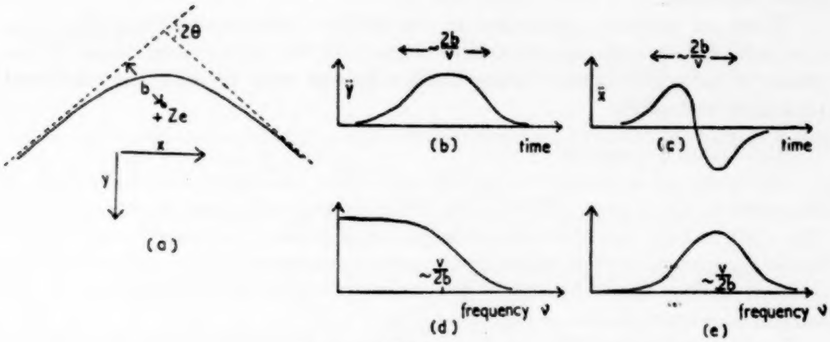

Fig. 1.

We shall want to calculate the radiation from the encounter. The electron, together with part of the charge on the stationary ion, constitutes an electric dipole of varying moment; this dipole may be regarded as a superposition of sinusoidally varying dipoles. If the moment is $p = p_0 \cos 2\pi\nu t$, then $\ddot{p}$ is simply the electronic charge times the appropriate Fourier component of the acceleration, and the power radiated in a medium of refractive index μ is

$$\frac{2}{3\mu c^3}\overline{(\ddot{p})^2} = \frac{1}{3\mu c^3} \text{ (mean squared amplitude of } \ddot{p}\text{).} \qquad (1)$$

Substitution of the appropriate accelerations (Fig. 1 (*b*), (*c*), (*d*), (*e*) in (1)), followed by integration over all collision parameters, should, if correctly performed, yield Elwert's classical approximation for a binary counter. However, the need to consider multiple collisions makes it more convenient to split the problem into two parts.

(i) "*Distant encounters*", *including multiple encounters*

In the sort of plasma we shall consider, the kinetic energy of a typical electron is very large compared to its mean negative potential energy in the field of an ion (i.e. $b_1 \gg b_4$), so that the electrons travel approximately in straight lines, with occasional sudden changes in direction due to close encounters with ions*.

So long as the electron moves almost uniformly in a straight line, the accelerations due to the electrostatic fields of the ions† of the plasma may be considered as small perturbations on a linear path. The acceleration experienced by the electron is then, to a good approximation,

$$\frac{e}{m}E(t),$$

where $E(t)$ is the field which would be experienced at time t by an electron moving through the plasma with uniform velocity.

* The condition $b_1 \gg b_4$ requires $N_i^{-1/3} \gg \frac{Ze^2}{\frac{3}{2}kT}$ that is $T \gg 10^{-3} Z N_i^{1/3}$ thus for hydrogen at 10^3 K (or hotter), $b_1/b_4 \geq 10^7 N_i^{-1/3}$ and the condition is valid by a large margin up to $N_i = 10^{12}$ cm^{-3}.

† Electron–electron collisions do not normally contribute to the radiation; see Section 7.

Using this approximation, the "multiple collisions" can be dealt with. We shall Fourier-analyse the field $E(t)$ due to a random distribution of ions, ignoring those ions which lie within a distance b_0 ($b_1 \gg b_0 \gg b_4$) of the electron's path; the value of $\ddot{p}$ corresponding to a Fourier component of $E(t)$ is obtained by multiplying by $e.(e/m)$, and can then be substituted in (1) to obtain the power radiated.

When an electron approaches an ion within a distance of the order of b_4, it is deflected through an appreciable angle, and the power contributed by the pulses of radiation emitted during close collisions must therefore be calculated separately and added.

(ii) "*Close encounters*"

The pulse of radiation emitted during a close encounter lasts for a time of the order of b/v or less. If $b/v \ll 1/\nu$, ν lies on the white part of the spectrum, Fig. 1 (*d*), and one can approximate to the actual pulse by an infinitely short pulse whose magnitude is determined by the total momentum change suffered by the electron during the encounter. Also, with $b/v \ll 1/\nu$ the contribution of the longitudinal accelerations is negligible (Fig. 1 (*e*)).

Now the condition $b/v \ll 1/\nu$ is equivalent to $b \ll b_2$. If $b_2 \ll b_1$, then almost all the encounters capable of emitting radiation of frequency ν are binary encounters, so that a further cut-off at a collision parameter b_1 need not be considered, and Elwert's formulae are clearly valid. If b_2 is greater than, or comparable with, b_1, then $b \ll b_1$ implies $b \ll b_2$; in all the cases we need to consider, $b < b_0 \ll b_1$, and hence $b \ll b_2$. Thus we can calculate the contribution of the close encounters ($b < b_0$) using the assumptions that

(*a*) the encounters are binary, since $b \ll b_1$;

(*b*) the pulses are infinitely short, since $b \ll b_2$.

4. *Calculations.*—We now calculate the radiation from the two classes of encounters separately, dividing the collision parameters into ranges above and below b_0. We shall then integrate over a Maxwellian distribution of speeds.

(i) *Distant encounters—*$b > b_0$

Consider an electron moving along the x axis, and an ion at $x=0$, $y=b$, $z=0$. The electrostatic potential V due to the ion at a point on the x axis is (4)

$$V = Ze(b^2+x^2)^{-1/2} = Ze\int_0^\infty \mathrm{Kh}_0(\xi b)\cos\xi x\, d\xi.$$

In the second expression, V has been Fourier-analysed into cosine components along the x axis with wave-numbers ξ. Kh is a Bessel function of imaginary argument, of the second kind, in the form used by Heaviside and by Jeffreys. The field components are therefore

$$E_x = -\frac{\partial V}{\partial x} = Ze\int_0^\infty \xi\,\mathrm{Kh}_0(\xi b)\sin\xi x\, d\xi,$$

$$E_y = -\frac{\partial V}{\partial b} = Ze\int_0^\infty \xi\,\mathrm{Kh}_1(\xi b)\cos\xi x\, d\xi,$$

$$E_z = 0.$$

Suppose we consider only the Fourier components of the field with wave-numbers ξ to $\xi+\delta\xi$. Then

$$E_x\,(\xi \text{ to } \xi+\delta\xi) = Ze\int_{\xi}^{\xi+\delta\xi} \xi\,\mathrm{Kh}_0(\xi b)\sin\xi x\,d\xi$$

$$= Ze\xi\,\mathrm{Kh}_0(\xi b)\sin\xi x\,d\xi\,.\frac{\sin\frac{1}{2}\delta\xi x}{\frac{1}{2}\delta\xi x},$$

and similarly

$$E_y\,(\xi \text{ to } \xi+\delta\xi) = Ze\xi\,\mathrm{Kh}_1(\xi b)\cos\xi x\,\delta\xi\,.\frac{\sin\frac{1}{2}\delta\xi x}{\frac{1}{2}\delta\xi x}.$$

A simple transformation of coordinates shows that for an ion at $(X, b\cos\phi, b\sin\phi)$ instead of $(0, b, 0)$, we should obtain

$$E_x\,(\xi \text{ to } \xi+\delta\xi) = Ze\xi\,\mathrm{Kh}_0(\xi b)\frac{\sin\frac{1}{2}\delta\xi(x-X)}{\frac{1}{2}\delta\xi(x-X)}\delta\xi\sin\xi(x-X),$$

$$E_y\,(\xi \text{ to } \xi+\delta\xi) = Ze\xi\,\mathrm{Kh}_1(\xi b)\frac{\sin\frac{1}{2}\delta\xi(x-X)}{\frac{1}{2}\delta\xi(x-X)}\cos\phi\,\delta\xi\cos\xi(x-X),$$

$$E_z\,(\xi \text{ to } \xi+\delta\xi) = Ze\xi\,\mathrm{Kh}_1(\xi b)\frac{\sin\frac{1}{2}\delta\xi(x-X)}{\frac{1}{2}\delta\xi(x-X)}\sin\phi\,\delta\xi\cos\xi(x-X).$$

The field actually experienced by the electron is the sum of fields E_i due to many ions; these lie at various distances X_i along the x axis, at various distances b_i from it, and various angles ϕ_i around it. We shall require the mean square of each component of the resultant field, for example

$$\overline{\{\sum E_{yi}\,(\xi \text{ to } \xi+\delta\xi)\}^2} = \sum\overline{E_{yi}^2(\xi \text{ to } \xi+\delta\xi)} + 2\sum\overline{E_{yi}\,(\xi \text{ to } \xi+\delta\xi)E_{yj}\,(\xi \text{ to } \xi+\delta\xi)}.$$

If the positions of the ions are uncorrelated, the field components due to the ith and jth ions are equally likely to have the same or opposite sign, and therefore the mean value of $\sum E_{yi}E_{yj}$ is zero. The mean number of ions in the typical volume element is $N_i\,.\,db\,.\,bd\phi\,.\,dX$, and hence

$$\overline{\{\sum E_{yi}\,(\xi \text{ to } \xi+\delta\xi)\}^2} = \sum\overline{E_{yi}^2}$$

$$= N_i\int_{b_0}^{\infty} Z^2e^2\xi^2\,\mathrm{Kh}_1^2(\xi b)b\,db\int_0^{2\pi}\cos^2\phi\,d\phi$$

$$\times\int_{-\infty}^{\infty}\frac{\sin^2\frac{1}{2}\delta\xi(x-X)}{\{\frac{1}{2}\delta\xi(x-X)\}^2}\cos^2\xi(x-X)\,dX(\delta\xi)^2$$

$$= Z^2e^2N_i\int_{b_0\xi}^{\infty}\mathrm{Kh}_1^2(\xi b)(\xi b)\,d(\xi b)\,.\,\pi\,.\,\tfrac{1}{2}\,.\,\frac{\pi}{\frac{1}{2}\delta\xi}\,.\,(\delta\xi)^2, \text{ since } \delta\xi \ll \xi$$

$$= \pi^2Z^2e^2N_i\delta\xi\int_{b_0\xi}^{\infty}\eta\,\mathrm{Kh}_1^2(\eta)\,d\eta \qquad (2)$$

and similarly

$$\overline{\{\sum E_{xi}\,(\xi \text{ to } \xi+\delta\xi)\}^2} = 2\pi^2Z^2e^2N_i\delta\xi\int_{b_0\xi}^{\infty}\eta\,\mathrm{Kh}_0^2(\eta)\,d\eta. \qquad (3)$$

The mean squared z component is equal to the mean squared y component.

The component of the field with wave-number ξ causes periodic acceleration of the electron with frequency $\nu=\xi v/2\pi$; the magnitude of the acceleration is

$$\frac{e}{m}\sum E_{xi}\,(\xi \text{ to } \xi+\delta\xi)$$

in the x direction, etc., and hence the mean power emitted according to (1) is

$$\frac{2}{3\mu c^3}(\overline{\ddot{p}_x^2}+\overline{\ddot{p}_y^2}+\overline{\ddot{p}_z^2}) = \frac{2}{3\mu c^3}\left\{\left(\frac{e^2}{m}\right)^2\overline{(\sum E_{xi}\,(\xi \text{ to } \xi+\delta\xi))^2}+\ldots\right\}$$

$$= \frac{2e^4}{3\mu c^3 m^2}2\pi^2 Z^2 e^2 N_i\,\delta\xi\int_{b_0\xi}^{\infty}\eta\{\mathrm{Kh}_0^2(\eta)+\mathrm{Kh}_1^2(\eta)\}\,d\eta$$

by (2) and (3)

$$= \frac{4\pi^2 Z^2 e^6 N_i\,\delta(2\pi\nu/v)}{3\mu c^3 m^2}\int_{b_0\,2\pi\nu/v}^{\infty}\eta\{\mathrm{Kh}_0^2(\eta)+\mathrm{Kh}_1^2(\eta)\}\,d\eta.$$

Integrating by parts,

$$\int\eta\,\mathrm{Kh}_1^2(\eta)\,d\eta=[\mathrm{Kh}_1(\eta)(-\eta\,\mathrm{Kh}_0(\eta))]-\int(-\eta\,\mathrm{Kh}_0(\eta))(-\mathrm{Kh}_0(\eta))\,d\eta$$

that is,

$$\int\eta\{\mathrm{Kh}_0^2(\eta)+\mathrm{Kh}_1^2(\eta)\}\,d\eta=[-\eta\,\mathrm{Kh}_0(\eta)\,\mathrm{Kh}_1(\eta)].$$

Since $\mathrm{Kh}_0(\eta)$ and $\mathrm{Kh}_1(\eta)$ both decrease exponentially as $\eta\to\infty$ we obtain: mean power radiated by an electron due to encounters with $b>b_0$

$$= \frac{8\pi Z^2 e^6 N_i\,\delta\nu}{3\mu c^3 m^2 v}\left(\frac{2\pi\nu}{v}b_0\right)\mathrm{Kh}_0\left(\frac{2\pi\nu}{v}b_0\right)\mathrm{Kh}_1\left(\frac{2\pi\nu}{v}b_0\right). \tag{4}$$

(ii) *Close encounters*—$b<b_0$

A binary encounter with collision parameter b deviates the path of an electron through an angle 2θ such that

$$\sin\theta=\{1+(bmv^2/Ze^2)^2\}^{-1/2}.$$

Hence the electron suffers a change in velocity of $2v\sin\theta$, and

$$\Delta\dot{p}=\int\ddot{p}\,dt=e\,.\,2v\sin\theta=2ev\{1+(bmv^2/Ze^2)^2\}^{-1/2}.$$

We now have to Fourier-analyse a sharp pulse in $\ddot{p}$, the magnitude of the pulse being given by the above value of $\int\ddot{p}\,dt$. Isolating the components in the frequency band ν to $\nu+\delta\nu$, we have

$$\ddot{p}\ (\nu \text{ to } \nu+\delta\nu)=2\Delta\dot{p}\int_{\nu}^{\nu+\delta\nu}\cos 2\pi\nu t\,d\nu$$

$$=2\Delta\dot{p}\,\delta\nu\,(\sin\pi\delta\nu t/\pi\delta\nu t)\cos 2\pi(\nu+\tfrac{1}{2}\delta\nu)t.$$

This represents an oscillation of frequency $\nu+\frac{1}{2}\delta\nu$ with a slowly varying amplitude $2\Delta\dot{p}\sin\pi\delta\nu t/\pi t$. By (1), the total energy radiated is

$$\frac{1}{3\mu c^3}\int_{-\infty}^{\infty}\left(2\Delta\dot{p}\frac{\sin\pi\delta\nu t}{\pi t}\right)^2 dt=\frac{1}{3\mu c^3}4(\Delta\dot{p})^2\,\delta\nu.$$

An electron moving with speed v has $2\pi b\,db\,vN_i$ encounters per second with collision parameters b to $b+db$; hence the close encounters ($b<b_0$) made by an electron with speed v contribute a mean power

$$\int_0^{b_0}\frac{4}{3\mu c^3}(\Delta\dot{p})^2 2\pi b\,db\,vN_i\,\delta\nu=\int_0^{b_0}\frac{4}{3\mu c^3}\frac{4e^2v^2}{1+(bmv^2/Ze^2)^2}2\pi b\,vN_i\,db\,\delta\nu$$

$$=\frac{16\pi Z^2 e^6 N_i}{3\mu c^2 m^3 v}\ln\left(1+\left(\frac{mv^2 b_0}{Ze^2}\right)^2\right)\delta\nu \tag{5}$$

in the frequency band ν to $\nu+\delta\nu$.

(iii) *Integration over a Maxwellian distribution of electron speeds*

The next step is to integrate both (4) and (5) over a Maxwellian distribution of speeds. For this purpose, neither expression is convenient, and the assumption $b_1 \gg b_0 \gg b_4$ must be used again to simplify them. In (4)

$$\frac{2\pi\nu}{v} b_0 = \frac{\sqrt{\overline{v^2}}}{v} \frac{b_0}{b_2}$$

and for all but the slowest electrons (about $(b_0/b_2)^3$ of the total) this number is $\ll 1$. Therefore one can approximate to the Bessel function in (4) by

$$\mathrm{Kh}_0(\eta) \simeq -\frac{2}{\pi}(\ln \tfrac{1}{2}\eta + \gamma); \qquad \mathrm{Kh}_1(\eta) \simeq \frac{2}{\pi\eta} \qquad (\gamma = 0{\cdot}577\ldots)$$

obtaining:

$$\text{Mean power} = \frac{32\pi Z^2 e^6 N_i \delta\nu}{3\mu c^3 m^2 v}\left(\ln\frac{v}{\pi\nu b_0} - \gamma\right). \tag{4a}$$

In (5),

$$\frac{mv^2}{Ze^2} b_0 = \frac{2v^2}{\overline{v^2}} \frac{b_0}{b_4}$$

which is $\gg 1$ for all but the slowest electrons (about $(b_4/b_0)^{3/2}$ of them), so that one may replace (5) by :

$$\text{Mean power} = \frac{32\pi Z^2 e^6 N_i \delta\nu}{3\mu c^3 m^2 v}\ln\left(\frac{mv^2}{Ze^2} b_0\right). \tag{5a}$$

Adding (4 *a*) and (5 *a*) and integrating over a Maxwellian distribution of speeds, we obtain the power radiated by the electrons in unit volume of the plasma:

$$\frac{32\pi Z^2 e^6 N_i \delta\nu}{3\mu c^3 m^2}\int_0^\infty \frac{1}{v}\left\{\ln\frac{mv^3}{\pi Ze^2\nu} - \gamma\right\}\frac{4}{\sqrt{\pi}} N_e \left(\frac{m}{2kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}}\, dv.$$

Using the formula

$$\int_0^\infty \ln\eta\, e^{-\eta}\, d\eta = -\gamma,$$

the integration can be performed, giving

$$\frac{32\pi Z^2 e^6 N_i N_e \delta\nu}{3\mu c^3 m(2\pi mkT)^{1/2}}\left\{\ln\frac{8k^3T^3}{\pi^2 Z^2 e^4 \nu^2 m} - 5\gamma\right\}. \tag{6}$$

To obtain the corresponding absorption coefficient, K, we note that the power radiated from unit volume is $K.8\pi kT\nu^2\delta\nu/c^2$, hence

$$K = \frac{8\pi Z^2 e^6 N_i N_e}{3\mu c(2\pi mkT)^{3/2}\nu^2}\left\{\ln\frac{8k^3T^3}{\pi^2 Z^2 e^4 \nu^2 m} - 5\gamma\right\}. \tag{7}$$

The "logarithmic factor"

$$\left\{\ln\frac{mv^3}{\pi Ze^2\nu} - \gamma\right\}$$

which occurs when (4 *a*) and (5 *a*) are added to obtain the mean power radiated by an electron is precisely the same as the factor occurring in Elwert's formula for the effective cross-section ((1), end of p. 480, formula for g'), and (7) is the formula which is obtained on integrating Elwert's results over a Maxwellian distribution.

On comparing (7) with Smerd and Westfold's formula

$$K = \frac{8\pi Z^2 e^6 N_i N_e}{3\mu c(2\pi mkT)^{3/2}\nu^2}\ln\left(1 + \left(\frac{4kT}{e^2} b_1\right)^2\right) \tag{8}$$

one finds that the essential difference is that in (7) b_1 has been replaced by b_2. That is to say, the upper limit to collision parameters does not occur at the mean interionic distance, but when the pulse of radiation becomes so long ($>b_2/v$) that it contains no appreciable Fourier component of frequency ν. Since (7) agrees precisely with the classical approximation to Elwert's binary collision formula, it appears that even in the multiple collisions, with $b > b_1$, the interaction between the electron and each ion contributes to the radiation as if it were a separate binary collision. This last fact may well be regarded as obvious if one considers that the radiation is proportional to $\ddot{p}^2$, and hence to the square of the field to which the electron is exposed, and that the mean square resultant of a number of vectors (the fields of the ions) is equal to the mean sum of the squares of the individual vectors (cf. Cohen, Spitzer and Routly (**5**) who make the same point in connection with electrical conductivity). The above calculation has nevertheless been set out in some detail, because the matter appears to have been in doubt for some time, and also because the formula (7) does not appear to have been published explicitly.

5. The numerical values of K given by the formula (7) do not differ drastically from those given by Smerd and Westfold's formula; for conditions of astrophysical interest the difference rarely exceeds 25 per cent. Perhaps the most interesting feature of (7) is that it predicts a slow decrease in the flux densities of thermal radio sources at high frequencies, rather than the perfectly white spectrum given by (8). For a nebula of protons and electrons at 10^4 °K and over the frequency range 300 Mc/s to 3000 Mc/s, the change would correspond to a spectral index (**6**) of $-0{\cdot}1$.

Owing to the large overlap between the regions of validity of the approximations used for close and distant encounters, formula (7) is a fairly precise one, and should be adequate for foreseeable radio-astronomical applications. The largest error is the neglect of Debye shielding (see Section 6); this introduces a fractional error of the order of $\nu_0/\nu \ln(b_2/b_4)$, which might amount to a few per cent near the turning point of a ray in the solar corona.

6. The crucial assumption in the calculation is the assumption that the positions of the ions are uncorrelated. This assumption requires critical examination, for the shielding of ions (originally investigated in the case of strong electrolytes) and the existence of collective phenomena show that it is, for many purposes, quite untrue.

The situation indicated by (6), that the mean power radiated is the same as if all electron–ion encounters were independent, may be pictured as follows: when an electron moves in the fields of several ions, these fields, in general, tend to cancel out, so that the radiation per encounter is reduced. There are, however, statistical fluctuations in charge density, and a region containing an excess of n ions will act like an ion of charge nZe on electrons passing near it. The power radiated is proportional to the square of the ionic charge, so that the cluster is n times as effective as the n ions individually. As shown in the previous section, the two effects cancel out exactly when the distribution of ions is random. In a real plasma, the probability that a particular kind of cluster of ions will occur is the product of its statistical weight and a factor $\exp(-\epsilon/kT)$ where ϵ is the electrostatic energy required to form the cluster. The average number of ions in a cube of side r is r^3N_i, and this number is subject to fluctuations of about $(r^3N_i)^{1/2}$; fluctuations greater than $5(r^3N_i)^{1/2}$ are very rare for statistical reasons alone.

The fluctuations in a real plasma are appreciably smaller than they would be in a random plasma if and only if the electrostatic energy

$$\epsilon = \tfrac{1}{2}\,(\text{charge})^2\,(\text{capacity})^{-1} \simeq \frac{\{Ze(r^3N_i)^{1/2}\}^2}{r} = Z^2e^2r^2N_i$$

is greater than or comparable with kT, that is, if

$$r^2 \geqslant kT/Z^2e^2N_i.$$

But

$$b_3 = \left(\frac{3kT}{4\pi N_i Z^2 e^2}\right)^{1/2} \simeq \frac{(\overline{v^2})^{1/2}}{2\pi\nu_0}$$

is the Debye shielding length; therefore we conclude that fluctuations on a smaller scale than the Debye length are unaffected by electrostatic forces; fluctuations on a larger scale than the Debye length are much smaller than they would be in a random distribution of ions. (Pines and Bohm (7) give a thorough discussion of fluctuations in a plasma.)

Since radio noise can only be emitted if the refractive index μ is real, the frequency ν radiated is always greater than the plasma frequency ν_0, and therefore $b_2 = (\overline{v^2})^{1/2}/2\pi\nu$ is always smaller than $b_3 = (\overline{v^2})^{1/2}/2\pi\nu_0$. But a charge concentration is only relevant to our problem if its diameter is of the order of b_2 or less; an electron would take longer than $1/\nu$ to pass by a larger charge concentration. The assumption of a random distribution of ions is therefore correct so far as the emission of radiation is concerned, with a possible error of the order of one in the logarithmic factor of (7) when ν is only a little greater than the plasma frequency ν_0.

Two minor assumptions which are implicit in the calculation may also be dealt with here. Firstly, it was assumed that the pulses of radiation, due to close encounters, occur at random intervals, and that the energies radiated by them may therefore be added. A collision with $b < b_0$ may be expected to occur once in $(\pi b_0^2 N_i)^{-1} = (b_1/b_0)^2(b_1/\pi)$ of electron trajectory. The exact positions of two ions so far apart could not be affected by their electrostatic repulsion, since their mutual potential energy $Z^2e^2(b_1/b_0)^{-2}(b_0/\pi)^{-1} = Z\tfrac{1}{2}kT\pi(b_0/b_1)^2(b_4/b_1)$ is much less than their thermal energy. To affect the relative phases of the radiation on frequency ν from the two pulses, ordering of the ions must have a precision of the order of b_2, and, since $b_2 < b_3$, the over-dispersion on a scale $> b_3$, or even collective phenomena due to non-thermal disturbances, are irrelevant. Secondly, it has been assumed, in calculating the radiation from distant encounters, that the electron's approximately linear trajectory continues indefinitely. In fact, the trajectory, and hence the wave-train radiated, is broken up into sections by close collisions. A rough estimate of this effect may be obtained by regarding these lengths of wave-train as uncorrelated in phase, as in the simple theory of the collision broadening of spectral lines; we then find that the spectrum of the radiation is smoothed over a frequency range of about

$$v(b_1/b_0)^{-2}(b_1/\pi)^{-1} \simeq 2\pi^2\nu(b_0/b_1)^2(b_2/b_1),$$

which is a very small fraction of ν. The unsmoothed spectrum, given by (6) contains no sharp resonances and the effect of the smoothing is therefore negligible.

7. *Radiation due to large concentrations of charge.*—The considerations of Section 6 lead naturally to a few comments on the possibility of radiation by

encounters between electrons and charge concentrations induced by non-thermal disturbances. This possibility was considered by Hoyle (**8**).

It was shown in Section 6 that, in a plasma in thermal equilibrium, statistical fluctuations in charge density are normal over regions less than b_3 in diameter, but fluctuations on a scale greater than b_3 are much less than in a random distribution of ions in space. Conversely, scales greater than the Debye length are the domain of collective phenomena; plasma oscillations and other collective phenomena on a scale smaller than the Debye length would be quickly damped out by diffusion. Now suppose that charge concentrations have been set up in a plasma by some external disturbance: these charge concentrations will be very much more effective in scattering electrons than single ions would be, owing to their greater charge. A charge concentration is at least b_3 in diameter, so that an electron takes a time $\geqslant b_3/v$ to pass by it, and might be expected to radiate a pulse of about this length. However, b_3/v is of the order of $(2\pi\nu_0)^{-1}$ for a typical electron of the plasma, so that the "pulse" contains no frequencies which can be propagated. Hoyle abandoned his suggestion for this reason. Radiation could only be emitted by means of a "space-charge concentration mechanism" if some electrons much faster than the thermal electrons were present.

It is not inconceivable that favourable conditions could occur for short periods in nature. In particular, Kahn (**9**) and Buneman (**10**) have shown theoretically that when two plasmas collide, the energy of mass motion is converted into chaotic particle motions via plasma oscillations of large amplitude and on a scale only a little larger than the Debye length. Complete charge separation on the scale of the Debye length is the best possible condition for radiation by the space-charge concentration mechanism. Buneman has even suggested that electrons with more than thermal velocities may be generated in the same process. It is also known from observation that streams of fast particles occur in various solar disturbances.

In estimating the radiation emitted (or the corresponding value of the absorption coefficient) one need only consider encounters with collision parameters greater than the radius r of a typical charge concentration, for an electron passing through the charge concentration does not experience the full Coulomb force corresponding to the total charge concentrated at the centre of the charge concentration. Also, the effective b_4 is always less than b_3, and $r \gtrsim b_3$, so that the approximate method (i) of Section 3 can be used; putting $b_0 = r$ in (4*a*) one obtains the mean power per fast electron (v is now the speed of the fast electrons, Ze the total charge of a charge concentration and N_i is the number of charge concentrations per unit volume (about $\frac{1}{4}r^3$ if they fill space)). The formula for the absorption coefficient, analogous to (7), is easily derived; it differs from (7) only in the interpretation of T, Ze, and N_i, and in the logarithmic factor, which is essentially $2\ln(b_2/b_4)$ in (7) and becomes $2\ln(b_2/r)$ for the present case (using the speed of the fast electrons in the definition of b_2).

The spectrum of radiation emitted by a space-charge concentration mechanism would be broad, and could not contain harmonically related frequency bands, such as are observed in some types of solar burst. Also, if a frequency n times higher than the plasma frequency is to be radiated, electrons with at least n times the speed of the thermal electrons must be present.

Normally, electron–electron encounters (and ion–ion encounters) can be neglected, for the two electrons suffer equal and opposite accelerations, and hence the dipole radiation from the two electrons interferes destructively. The

quadrupole radiation from the pair of electrons is important only if their separation b is comparable with the wave-length of the radiation; but the radiation emitted in the encounter extends up to a frequency v/b, and hence down to a wave-length of $(c/v)b$, which is $\gg b$ unless v is comparable with c. For relativistic electrons, the electron–electron encounters might contribute appreciably to the radiation.

The absence of radiation from electron–electron encounters has one entertaining consequence. Suppose a fast electron moves in a plasma in which there are concentrations of ions, but that these concentrations also have enough electrons associated with them to make them substantially neutral. According to the above argument, the presence of the electrons should not affect the amount of radiation emitted by the fast electron, but common sense insists that the electron cannot be scattered significantly by a blob of neutral plasma. The correct physical picture in this case must be that the fast electron is virtually undeviated, but it repels all the electrons in the blob of high density, and these electrons therefore radiate coherently. In other words, the fast electron temporarily polarizes the entire blob. Thus real charge concentrations are not necessary for radiation by the "space-charge concentration mechanism"; irregularities of density, such as might be produced in shock waves or sound waves, would be just as effective, if they could be formed on a sufficiently small scale.

However, in astrophysical examples, the mean free path (the smallest scale of sound waves) is very much larger than the Debye length, so that this version of the mechanism is unlikely to be common in nature. If a magnetic field permeates the plasma, the smallest possible scale of "sound" waves is the Larmor radius, but even then a fairly large field (such as that over a sunspot) is needed to make "useful" irregularities possible; in a field determined by equipartition between magnetic and thermal energy, the Larmor radius would be c/ν_0, the "plasma wave-length", which is much too large.

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Cavendish Laboratory,
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References

(1) G. Elwert, *Z. für Naturforschung*, **3A**, 477, 1948.
(2) S. F. Smerd and K. C. Westfold, *Phil. Mag.*, **40**, 831, 1949.
(3) I. A. Gaunt, *Proc. Roy. Soc.* A, **126**, 654, 1930.
(4) H. and B. S. Jeffreys, "Methods of Mathematical Physics", C.U.P., 2nd Edition, pp. 590–591.
(5) R. S. Cohen, L. Spitzer, and P. McR. Routly, *Phys. Rev.*, **80**, 230, 1950.
(6) G. R. Whitfield, *M.N.*, **117**, 680, 1957.
(7) D. Pines and D. Bohm, *Phys. Rev.*, **85**, 338, 1952.
(8) F. Hoyle, *Nature*, **172**, 296, 1953.
(9) F. D. Kahn, *J. Fluid Mechanics*, **2**, 601, 1957.
(10) O. Buneman, "The dissipation of currents in ionised media" accepted for publication in *Phys. Rev.*).

A SPECTRAL ANALYSIS OF THE RADIO SOURCES IN CYGNUS X AT 1390 MC/S AND 408 MC/S

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Summary

A high resolution survey of the Cygnus X region was made at 408 Mc/s using the 250 ft paraboloid at Jodrell Bank. The results were combined with the high resolution surveys at 1390 Mc/s of Drake and Westerhout to investigate the spectral characteristics of the individual radio sources in Cygnus X. The results support the previous conclusions that the emission is from optically thin H II regions except for the strongest source in the region, the γ Cygni source, which was found to have a non-thermal component.

Introduction.—All surveys of the radio emission of the Cygnus X region have indicated that the origin of the radiation is in an extended H II region. Piddington and Minnett (1) first showed that the source had a thermal spectrum. Davies (2) has confirmed this theory and also concluded that the source is located in the 2nd spiral arm of the galaxy. Recent high resolution surveys at 1390 Mc/s by Drake (3) and Westerhout (4) have resolved the region into a large number of sources.

The 250 ft paraboloid at Jodrell Bank (5) was used to survey the Cygnus X region at a frequency of 408 Mc/s. As the beam size was the same as that used by Drake at 1390 Mc/s, a detailed investigation of the spectral characteristics of the individual sources was possible.

1. *Theory*

(*a*) *Flux measurements at* 408 *Mc/s using the* 250 *ft paraboloid.*—The terminology in this section is that given by Seeger, Westerhout and van de Hulst (6).

The full beam pattern of the radio telescope was obtained using the Cassiopeia radio source, from which the effective solid angle Ω' of the beam was calculated to be 0·68 sq. degrees. Ω' is given by the expression

$$\Omega' = \int_{\text{full beam}} f(\theta, \phi)\, d\Omega$$

in which $f(\theta, \phi)$ is the power received in the direction (θ, ϕ) normalized by putting $f = 1$ at the centre of the beam. The full beam is defined by a circle with diameter 5 times the half-power diameter of the beam.

The sensitivity was checked daily by measurement of the deflection D_0 produced by the 14N5A radio source. This source has a flux density S_0 of $47 \times 10^{-26}\, \mathrm{wm^{-2}\,(c/s)^{-1}}$ (7).

The flux density S of any distributed region bounded by the limits (α_1, α_2) in right ascension and (δ_1, δ_2) in declination is then given by

$$S = S_0 \cdot \frac{\int_{\alpha_1}^{\alpha_2}\int_{\delta_1}^{\delta_2} D(\alpha, \delta)\, d\alpha\, d\delta}{\Omega' D_0}$$

where $D\,(\alpha, \delta)$ is the deflection at position (α, δ).

From these flux measurements the brightness temperature T_b of a uniformly intense region is derived by means of the standard formula (**8**)

$$T_b = \frac{S\lambda^2}{2k\Omega_s}$$

where Ω_s is the solid angle subtended by the source. Thus the brightness temperature is related to the deflection D by the expresssion

$$T_b = \frac{S_0\lambda^2}{2k\Omega'} \cdot \frac{D}{D_0} \cdot$$

(*b*) *Spectral analysis.*—In comparing surveys of regions at different frequencies with telescopes of different size, the brightness temperature of an extended region depends only on the spectral characteristics of the emission. For point sources the brightness temperature depends not only on the spectral characteristics of the source but is proportional to the size and diffractive efficiency of the aerial.

Seeger, Westerhout and van de Hulst (**6**) have shown that for a telescope of effective solid angle Ω' an increase of flux from a point source of $2k\Omega/\lambda^2$ at the centre of the beam is equivalent to a 1 °K increase in T_b over the full beam.

Two spectral classes of radio sources are commonly recognized, i.e., those with a thermal spectrum and those with a non-thermal spectrum. For thermal sources the radio brightness (flux density per steradian) has a spectral index which may range from +2 for optically thick regions to 0 for optically thin regions. For non-thermal sources the spectral index usually lies between 0 and −1.

A comparison of the spectral characteristics of the sources in Cygnus X is made in the following sections using the 20 cm survey of Drake (**3**) and the present 75 cm survey. The effective solid angle of the aerial beams was the same. Table I gives the ratios of brightness temperature to be expected in these two surveys for various classes of radio source.

Table I

Class of source	Spectral index: Brightness flux/ster. density	Spectral index: T_b	$\frac{T_{408}}{T_{1390}}$
Extended optically thick thermal	+2	0	1
Extended optically thin thermal	0	−2	12
Extended non-thermal	0	−2	12
	−1	−3	40
Point non-thermal	0	−2	12
	−1	−3	40

2. *Results.*—Fig. 1 shows the 408 Mc/s radio brightness contours of the Cygnus X region. Fig. 2 is a typical scan at constant declination $+43^\circ 45'$. The sources are listed under their Westerhout numbers (4), together with positions and peak brightness temperatures, in Table III. The criterion used in the recognition of a source is that it should be possible to draw a closed contour comparable with the beam width around the point. All the temperatures have been measured above a reference region (R.A. $20^h 58^m$, Dec. $+39^\circ$) which is 5° from the galactic plane.

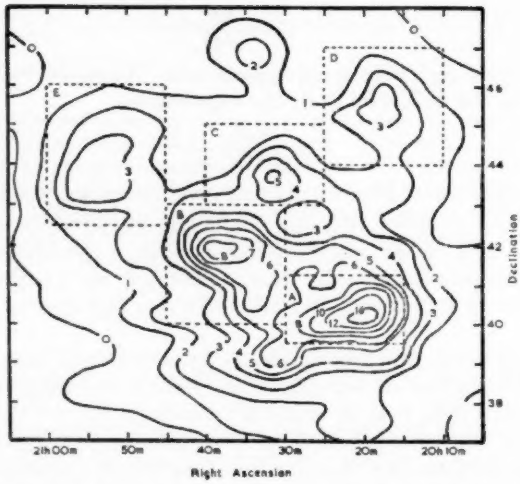

FIG. 1.—408 *Mc/s radio contours of the Cygnus X region, indicating the brightness temperature above a chosen reference region R.A.* $20^h 58^m$, *Dec.* $+ 39^\circ$. *The contour brightness temperature unit is* 28 °*K. The co-ordinates are for epoch* 1950.0. *The dotted rectangles, A, B, C, D and E, are the regions selected for flux comparison measurements* (*Section* 3).

FIG. 2.—*A typical drift curve through the Cygnus X region at Dec.* $+43^\circ 45'$.

It is assumed that the temperature of this reference region is representative of the background temperature for the whole Cygnus X region. Scans at declination $+49^\circ$ above the Cygnus X region show that the temperature at a point on the galactic plane is not greater than 20°K above the reference point. It is not likely therefore that the background flux contributes more than 10 per cent to our measurements of flux density.

3. *Comparisons of 20 cm and 75 cm surveys.*—The total integrated flux density from a region extending from R.A. $20^h 08^m$ to $20^h 47^m$ and Dec. 37° to 45° is 8600×10^{-26} w.m.$^{-2}$ (c/s)$^{-1}$ at 408 Mc/s. Within the same limits and above the same reference region the flux density was measured from Drake and Westerhout's maps and found to be 6200×10^{-26} and 7400×10^{-26} w.m.$^{-2}$ (c/s)$^{-1}$ respectively. These flux density values are in good agreement thus supporting the previous theories that the emission has a thermal origin in optically thin H II regions.

The spectral characteristics of the individual features were investigated both by integrated flux density measurements of five selected areas of the map (Fig. 1 and Table II) and by comparison of the brightness temperatures of the sources (Table III).

TABLE II

Comparison at 1390 Mc/s and 408 Mc/s of the flux density from the 5 selected regions shown dotted in Fig. 1

Region	R.A. limits	Dec. limits	408 Mc/s $\times 10^{-26}$ wm^{-2} (c/s)$^{-1}$	1390 Mc/s Drake $\times 10^{-26}$ wm^{-2} (c/s)$^{-1}$	1390 Mc/s Westerhout $\times 10^{-26}$ wm^{-2} (c/s)$^{-1}$
	h m h m	° ′ ° ′			
A	20 15 to 20 30	+39 30 to +41 15	1900	1400	1400
B	20 30 to 20 45	+40 to +43	1800	1800	1700
C	20 25 to 20 40	+43 to +45	700	500	600
D	20 10 to 20 25	+44 to +47	600	...	500
E	20 45 to 21 00	+42 30 to +46	1000	...	1000

TABLE III

List of positions and peak brightness temperature of radio sources measured at 408 Mc/s together with spectral indices obtained from comparison with Drake (3) *and Westerhout's* (4) *1390 Mc/s surveys*

Westerhout No.	R A. (1950)	Dec. (1950)	T_b peak 408 Mc/s °K	T_b peak 1390 Mc/s	$\frac{T_{408}}{T_{1390}}$	Spectral index	Remarks
	h m s	° ′					
63	20 17 30	45 22	104	8	13	−2·1	From Westerhout (4)
64	20 21 10	37 28	70	7	10	−1·9	From Westerhout. Poor positional agreement.
66	20 20 00 ±04s	40 10 ±2′	500	28	18	−2·4	γ Cygni source.
69	20 30 40	39 12	170	14	12	−2	
70	20 31 30	43 44	144	11	13	−2·1	
71	20 33 50	46 47	64	5	13	−2·1	From Westerhout.
73	20 33 50	40 58	178	18	10	−1·9	
75	20 38 40	41 55	250	22	11	−2	
76	20 40 20	39 00	77	6	13	−2·1	
77	20 42 40	40 52	70	5	14	−2·1	
80	20 53 30	43 44	108	10	11	−2	From Westerhout.

The probable error of source position (except for No. 66) is 5 min. of arc.

From the equal contributions to the flux at the two frequencies from four (B, C, D and E) of the five regions it is seen that most of the sources are optically thin H II regions. This is supported also by the marked similarity of the contours at the two frequencies. The ratios of brightness temperatures (Table III) show that T_b varies as f^{-2} for all the sources with the exception of No. 66, which lies in Region A (Fig. 1).

These measurements show that the spectrum of the Cygnus X region is thermal throughout, apart from the strongest source No. 66 (the γ Cygni source).

4. *The nature of the γ-Cygni source.*—From the previous section it is seen that the ratio of brightness temperatures for the No. 66 source is 18, i.e. $T_b \alpha f^{-2 \cdot 4}$ which cannot be accounted for by a thermal origin. Therefore it seems likely that at least part of the flux from this source has a non-thermal origin.

A model is proposed for this source which consists of a non-thermal source, flux density αf^{-1} and an optically thin thermal source with brightness temperature αf^{-2}. Table IV gives the contributions to the brightness temperature at the two frequencies from the thermal and non-thermal components in the γ Cygni source.

TABLE IV

1390 Mc/s		408 Mc/s	
Thermal	Non-thermal	Thermal	Non-thermal
22°K	6°K	250°K	250°K

If there is a non-thermal point source in the position of No. 66 which contributes 250°K to the brightness temperature, then its flux density at 408 Mc/s is 230×10^{-26} w.m.$^{-2}$ (c/s)$^{-1}$. This represents a practical minimum value for the flux density from the non-thermal component since an assumption of a less steep spectrum or finite optical depth in the thermal region would increase this flux density. If the non-thermal component is appreciably extended, this value once again represents a minimum value for its flux density.

Interferometer measurements at 158 Mc/s on baselines of 220λ and 2100λ show that there is no point source greater than 1 per cent of Cygnus A at the position of γ Cygni*. Therefore it seems that there is no source less than 20′ of arc with flux density greater than 50×10^{-26} w.m.$^{-2}$ (c/s)$^{-1}$ at 408 Mc/s. The size of the γ Cygni source measured in this present survey is 1°. It appears that the non-thermal source has a diameter between $\frac{1}{3}$° and 1°.

From Table II there is an excess flux density at 408 Mc/s of 500×10^{-26} w.m.$^{-2}$ (c/s)$^{-1}$ in Region A which includes γ Cygni. If this is due to the non-thermal source, then to contribute 250°K to the brightness temperature the source size would be about 1°.

5. *Conclusion.*—The spectral characteristics of the individual sources of the Cygnus X region have been studied at two frequencies 1390 Mc/s and 408 Mc/s. The results indicate that the spectrum of most sources is that of optically thin H II regions. Thus the emission measures derived by Drake and Westerhout from their 1390 Mc/s surveys are justified. It has also been shown that the spectrum of the strongest source in the region is non-thermal and on the basis of the relative contributions assigned in the model to thermal and non-thermal components at 1390 Mc/s, it is necessary to reduce the emission measures for this source

* Data kindly supplied by Dr H. P. Palmer.

from 25 000 to 20 000 cm^{-6} pc. Rishbeth (9) tentatively associates an extended non-thermal source in the Orion region with an extensive dark nebula. It is interesting to note that the γ Cygni source also lies in an unusual structure of dark nebulosities (3).

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References

(1) Piddington, J. H. and Minnett, H. C., *Aust. J. Sci. Res.*, **5A**, 17, 1952.
(2) Davies, R. D., *M.N.*, **117**, 663, 1957.
(3) Drake, F. D., *Paris Symposium on Radio Astronomy*, ed. R. N. Bracewell, Stanford Univ. Press, 1959 (in press).
(4) Westerhout, G., *B.A.N.*, **14**, 215, 1958.
(5) Lovell, A. C. B., *Nature*, **180**, 60, 1957.
(6) Seeger, C. L., Westerhout, G., and van de Hulst, H. C., *B.A.N.*, **13**, 89, 1956.
(7) Seeger, C. L., Westerhout, G. and Conway, R. G., *Astrophys. J.*, **126**, 585, 1957.
(8) Pawsey, J. L., and Bracewell, R. N., *Radio Astronomy*, Clarendon Press, Oxford, 1955.
(9) Rishbeth, H., *M.N.*, **118**, 591, 1958.

THE SPECTRUM OF THE GALACTIC RADIO EMISSION

C. H. Costain

(Communicated by M. Ryle)

(Received 1959 August 24)

Summary

Absolute measurements of the cosmic background radiation have been made at wave-lengths of 1·7 m and 7·9 m. Scaled aerials were used in conjunction with thermally-calibrated receivers in order to make an accurate determination of the spectrum of the emission. Observations by Baldwin at a wave-length of 3·7 m have been smoothed to provide a comparison at an intermediate frequency.

The spectral index, x, defined by $T_A \propto \lambda^x$, was found to be $2{\cdot}37 \pm 0{\cdot}04$. There are no significant departures from this value, measured over solid angles of approximately 0·2 steradians, except for regions on the galactic plane within 60° of the galactic centre.

The possibility of detecting the extra-galactic component of the integrated emission by means of its spectral index is discussed and upper limits are derived.

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1. *Introduction.*—The spectrum of the cosmic radio emission and its spatial variation are of utmost importance in investigations of the theories of origin of the radiation. Absolute measurements with large and complex aerial systems have proved very difficult. A comparison of such measurements made at different frequencies by various workers yields a wide range of values for the spectral index.

A more promising approach is to use identical aerial systems at various frequencies (**1**). Direct comparison of aerial temperatures may then be made and the problem is reduced to a determination of receiver sensitivity. The use of scaled aerials necessarily limits the resolving power that can be easily obtained. It does, however, permit comparison of the more extended components of the galactic emission, and the mean temperatures derived may be used to calibrate larger instruments.

This paper contains the results of measurements of the brightness of the northern sky at wave-lengths of 1·7 and 7·9 m. Corner reflector aerials with reception patterns of $15° \times 40°$ were used.

The magnitude of an isotropic extra-galactic component of the background has been discussed in relation to a number of cosmological models (**2, 3**). Observationally, such a component is indistinguishable from the outer shell of the galactic halo if it has the same spectrum; an upper limit may be placed on its magnitude from considerations of symmetry (**4**). If, however, its spectrum differs from that of the galactic emission, the present observations allow somewhat finer limits to be set than those determined above.

2. *Apparatus*.—The aerials used in these observations were of the 90° corner reflector type with sides $1 \cdot 1\lambda$ by 4λ. Each contained 4 full-wave dipoles $0 \cdot 45\lambda$ from the apex. They were mounted on an East–West line, $0 \cdot 2\lambda$ above the ground and could be tipped in declination. The reception pattern of each aerial covered 15° in right ascension and 40° in declination. They were matched prior to the observations to give a S.W.R. < 1·05.

Phase-switching receivers (**5**) were used at both frequencies. A 'hybrid-ring' network provided the two correlated inputs required for this type of receiver. It is shown in the next section that this is effectively a Dicke (**6**) system but has important advantages when comparing noise sources of differing impedances.

Two sources of noise power were used to measure the sensitivity of the equipment: a temperature-limited diode source for daily measurements, and a thermal noise source for the absolute calibrations. The thermal sources used have been described elsewhere (**1**).

3. *Observational method*.—The arrangement of the apparatus is shown in Fig. 1.

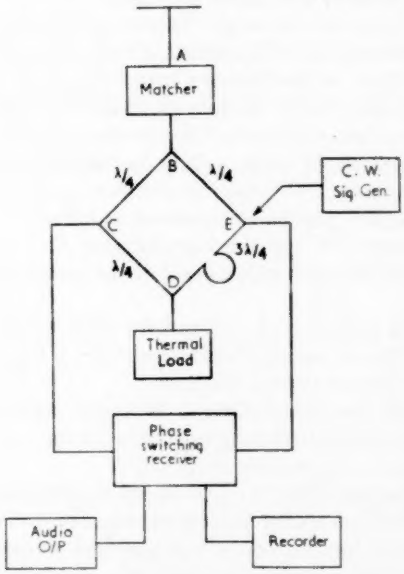

FIG. 1.—*A schematic diagram of the apparatus.*

The aerial or noise source is connected at A through the matcher to B. The comparison source at D is a thermal load immersed in water in a thermos flask and is maintained at room temperature. The matcher is adjusted so that the impedances presented to B and D on the network are equal.

Let the noise powers at B and D be represented by their effective temperatures T_B and T_D, and that of the receiver by T_R. The temperature at the input of the receiver in the 'in-phase' position of the switch is given by

$$T_R + \left(\frac{T_B}{2} + \frac{T_D}{2}\right) + \left(\frac{T_B}{2} - \frac{T_D}{2}\right) = T_B + T_R. \tag{1}$$

The corresponding expression for the anti-phase position of the switch is

$$T_R+\left(\frac{T_B}{2}+\frac{T_D}{2}\right)-\left(\frac{T_B}{2}-\frac{T_D}{2}\right)=T_D+T_R. \qquad (2)$$

The output of the phase-switching receiver is proportional to the difference between (1) and (2), i.e. is proportional to

$$T_B-T_D.$$

The system is therefore equivalent to the more straightforward Dicke system where the receiver is switched to the aerial and the comparison source in turn. Its advantage lies in the fact that a C.W. signal (at the centre frequency of the system) introduced at E, appears at C attenuated by an amount proportional to Z_B-Z_D, where Z_B and Z_D are the impedances presented to the network by the matcher and thermal load respectively. With a carefully constructed network, the matcher may be adjusted to give an insertion loss between C and E of the order of 80 db. The matched condition, when the various sources are interchanged, may therefore be reproduced with great precision.

The arguments above refer to a single discrete frequency, whereas the observations are made of noise signals with systems of finite bandwidth. It is essential that the matching be done at the effective centre frequency of the noise band; otherwise the receiver output may be dependent on the impedance of the noise source. This condition may be checked by transforming the source impedance through an arbitrary length of cable. After re-matching, the receiver output should be the same as before.

The procedure adopted may be summarized as follows:

(i) A large modulated C.W. signal is introduced at E.

(ii) The matcher is adjusted with the aid of the audio-detector to reject the modulated signal.

(iii) The receiver is reconnected to E and the recorder output noted.

(iv) Steps (i) to (iii) are repeated with an additional length of cable (usually $\lambda/4$) between the noise source and the matcher.

(v) If necessary, the frequency of the C.W. signal generator is adjusted and the procedure repeated until introducing an arbitrary length of cable and rematching produces no change in the receiver output.

The aerials were moved in 20° steps between declinations +20° and +80°. At each setting a continuous record was taken over a period of 24 hours. Each run was repeated until at least two records, consistent throughout to better than 5 per cent, were obtained. Daily measurements of sensitivity and zero were made, some through the aerial cable in order to determine the cable attenuation. A comparison of the diode noise source with a thermal load had been made by F. G. Smith prior to this set of observations. This comparison was repeated by the author.

4. *Method of analysis*.—The records selected as described above were sampled at hourly or half-hourly intervals. The measurements of cable attenuation, zero-level, and sensitivity were used to convert the record deflections to equivalent aerial temperatures. Where a number of values were available, the mean was computed. The results are shown in Figs. 2, 3, 4 and 5.

The results of a survey by Baldwin (7) at 3·7 m with a system of higher resolving power were smoothed over 15° × 40° in order to permit a comparison with the present observations. The smoothed scans are also shown in Figs. 2–5.

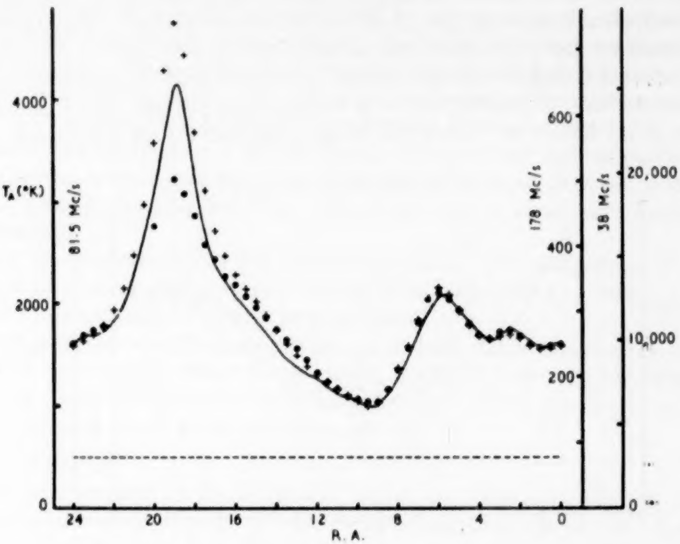

FIG. 2.—*The variation of aerial temperature on Dec. +20° at wave-lengths of 1·7 m (continuous line), 3·7 m (crosses) and 7·9 m (filled circles).*

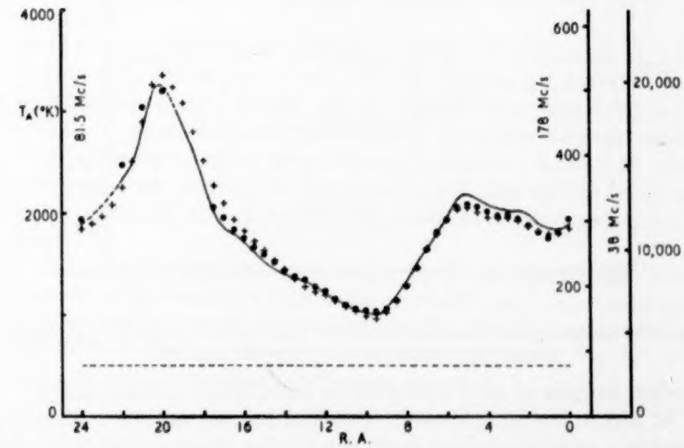

FIG. 3.—*The variation of aerial temperature on Dec. +40° at wave-lengths of 1·7 m (continuous line), 3·7 m (crosses) and 7·9 m (filled circles).*

The spectral index was then determined by finding the scaling factor which gave the best agreement between the temperature curves obtained at the three frequencies. No further adjustments of temperature scales have been made.

The effects of aerial smoothing and reception through subsidiary maxima may be neglected when comparing the 1·7 m and 7·9 m observations. The smoothing function used for the 3·7 m observations contained no side lobes. The highest temperatures are therefore too high and the lowest very slightly too low in relation to the other curves. This effect may be as much as 5–10 per cent near R.A. $19^h 00^m$ on the curve for Dec. + 20°, but it will be much smaller for the other parts of the sky.

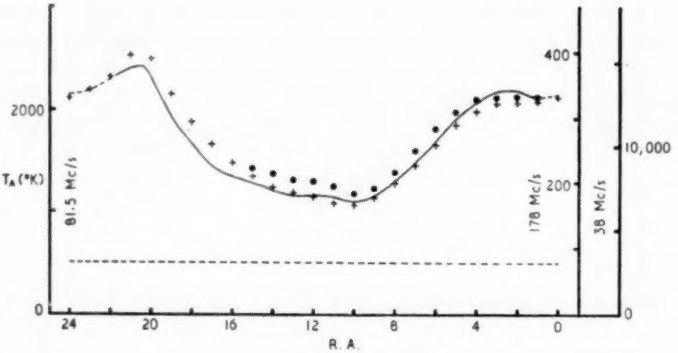

FIG. 4.—*The variation of aerial temperature on Dec. +60° at wave-lengths of 1·7 m (continuous line), 3·7 m (crosses), and 7·9 m (filled circles).*

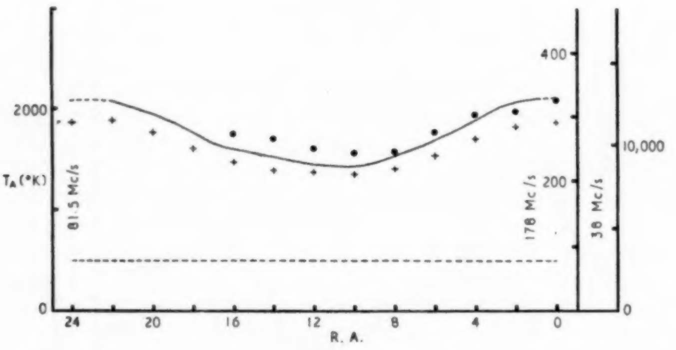

FIG. 5.—*The variation of aerial temperature on Dec. +80° at wave-lengths of 1·7 m (continuous line), and 3·7 m (crosses), and 7·9 m (filled circles).*

The other sources of error that must be considered are listed below:

(i) The loss in the cable between the aerial and the receiver. This quantity was measured a number of times and a mean value obtained with an accuracy of 1 per cent.

(ii) Variation in the temperature of the cable with respect to the reference thermal load. With the cable attenuations concerned this effect would be of the order of 1 per cent and has been neglected.

(iii) The reading accuracy of the records. This is about 1 per cent for one individual record and the error of the mean is therefore insignificant.

(iv) Fluctuations in receiver sensitivity and zero level. The receiver sensitivity was maintained at a nearly constant level by means of an AGC system. The 7·9 m observations were often seriously affected by daytime interference from R.A. $17^h 00^m$. As mentioned above, records showing departures from the mean curve of more than 5 per cent were discarded so the relative accuracy of most of the scans should be about 2–3 per cent. Where bad observing conditions persisted (00^h to 06^h, Dec. $+40°$ at 1·7 m; $17^h 00^m$ to $24^h 00^m$, all declinations at 7·9 m) wider tolerances had to be accepted and the accuracy is of the order of 5 per cent. This is also true for Dec. $+80°$ where only 2 scans were made at each wave-length.

(v) The accuracy of the noise power standard. The calibrations of the diode noise source by the author agreed with those of F. G. Smith, to within 2 per cent. An absolute accuracy of 5 per cent is assumed.

It is estimated that the temperature ratios obtained from most of the individual points are accurate to about 10 per cent. Where a large number of points are used, the accuracy improves to about 7 per cent, which is the present estimate of the absolute accuracy of the whole system.

5. *Discussion.*

(i) *The spectrum of the background radiation.*—The most obvious feature of the curves shown in Figs. 2–5 is the excellent agreement at all three frequencies over most of the sky. It is clear that the components responsible for a very high proportion of the integrated emission all have the same spectral variation. Only on the galactic plane at Dec. $+20°$, where the effects of H II clouds may be expected to become important, is there any clear evidence for variation in the spectral index. The observations therefore confirm the conclusions of Adgie and Smith (**1**) based on observations over the same frequency range with aerials of lower resolving power.

Another feature of probable significance is the lower intensity at 1·7 m from R.A. $12^h 00^m$–$16^h 30^m$ on Dec. $+20°$. In this region, a substantial portion of the radiation comes from a narrow belt of emission extending from the galactic plane near $l = 0°$, up towards the galactic pole. A similar reduction of intensity at 175 Mc/s is also shown in the observations of Adgie and Smith (**1**). Additional support for a steeper spectrum in this region is provided by the survey of Kraus and Ko (**8**), where the feature is much less prominent than at lower frequencies (**7**, **9**, **10**).

The absence of a significant variation in the spectral index over most of the sky allows the observational errors to be averaged out. The accuracy of the spectral index is then determined primarily by the absolute intensity scales. All points $00^h 00^m$ to $17^h 00^m$ in Figs. 2–5 have been replotted in Fig. 6. The value of the spectral index determined from these points is

$$x = 2{\cdot}37 \pm 0{\cdot}04.$$

This value is also consistent with the 3·7 m observations.

(ii) *The extra-galactic component.*—The determination of the integrated emission from extra-galactic sources and hence the average emission/unit volume of extra-galactic space is of considerable interest, both as a possible means of distinguishing between different cosmological models (**2**, **3**) and as evidence on the nature of radio stars (**11**). The present observations now make it possible to obtain additional information.

Part of the extra-galactic emission is due to normal galaxies, whose spectra may be expected to be similar to our own; observations of the Andromeda nebula (**12**, **13**) have given figures close to that found in the present observations. Part will however be due to the more powerful extra-galactic sources; all of those whose spectra have been measured have been found to have an emission which varies as $\lambda^{0 \cdot 8}-\lambda^{1 \cdot 2}$. The integrated emission will therefore have a component whose brightness temperature will lie in the range $\lambda^{2 \cdot 8}-\lambda^{3 \cdot 2}$. An estimate of the upper limit of this latter component is of great interest (**11**).

Whilst the incidence of radiation from normal galaxies will appear as an equal displacement at all wave-lengths of the zero level of the galactic emission (the dotted line in Figs. 2–5 indicates the upper limit of the total extra-galactic emission derived by Baldwin (**4**)), extra-galactic emission having a steeper spectrum will give rise to unequal displacements at different wave-lengths.

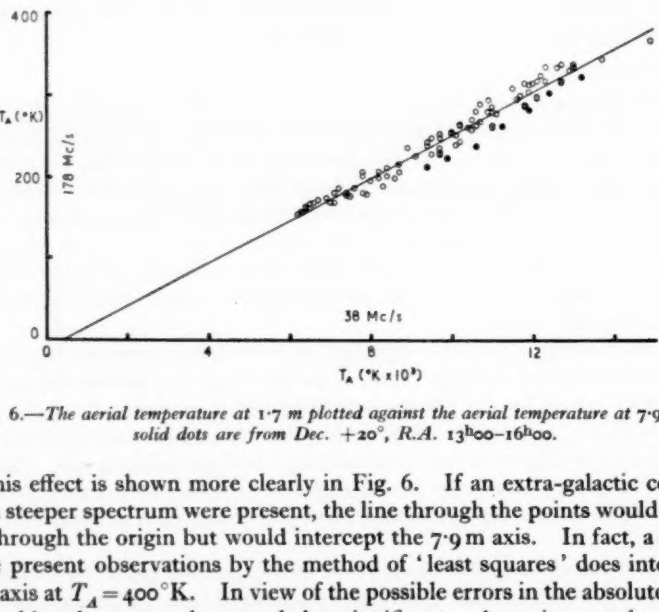

FIG. 6.—*The aerial temperature at 1·7 m plotted against the aerial temperature at 7·9 m. The solid dots are from Dec. +20°, R.A. 13h00–16h00.*

This effect is shown more clearly in Fig. 6. If an extra-galactic component with a steeper spectrum were present, the line through the points would no longer pass through the origin but would intercept the 7·9 m axis. In fact, a line fitted to the present observations by the method of 'least squares' does intercept the 7·9 m axis at $T_A = 400$ °K. In view of the possible errors in the absolute intensity scales, this value cannot be regarded as significant. A maximum value of 900 °K at 7·9 m has been determined for the excess emission. Table I shows the maximum possible magnitude of any component of the integrated extra-galactic radiation due to sources with a spectral index in the range $S \propto \lambda^{0 \cdot 8}$ to $S \propto \lambda^{1 \cdot 2}$, corresponding to brightness temperatures proportional to $\lambda^{2 \cdot 8}$ to $\lambda^{3 \cdot 2}$.

TABLE I

Extra-galactic contribution ($\lambda = 3 \cdot 7$ m)	Spectral index
< 240 °K	2·8
< 155 °K	3·0
< 110 °K	3·2

The contribution from normal galaxies has not been included. The last value corresponds to the mean spectral index determined by Whitfield (**12**) for 63 radio sources at galactic latitudes greater than 10°.

I would like to express my thanks to Dr F. G. Smith, who was largely responsible for the development of the experimental technique and who also assisted with the 7·9 m observations, and to Dr J. Baldwin who assisted with the 1·9 m observations. My thanks are due also to Professor M. Ryle for helpful discussion. I am indebted to the National Research Council of Canada for a Special Scholarship.

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References

(**1**) R. Adgie and F. G. Smith, *Observatory*, **76**, 181, 1956.
(**2**) J. R. Shakeshaft, *Phil. Mag.*, **45**, 1136, 1954.
(**3**) W. Priester, *Z. Astrophys.*, **34**, 295, 1954.
(**4**) J. E. Baldwin, *M.N.*, **115**, 690, 1955.
(**5**) M. Ryle, *Proc. Roy. Soc.*, A, **211**, 351, 1952.
(**6**) R. H. Dicke, *Rev. Sci. Instruments*, **17**, 268, 1946.
(**7**) J. E. Baldwin, *M.N.*, **115**, 684, 1955.
(**8**) H. C. Ko and J. D. Kraus, *Sky and Telescope*, **16**, 160, 1957.
(**9**) J. S. Hey, S. J. Parsons and J. W. Phillips, *Proc. Roy. Soc.*, A, **192**, 425, 1948.
(**10**) J. G. Bolton and K. C. Westfold, *Aust. J. Sci. Res.*, **3A**, 19, 1950.
(**11**) M. Ryle, *Proc. Roy. Soc.*, A, **248**, 289, 1958.
(**12**) G. R. Whitfield, *M.N.*, **117**, 680, 1957.
(**13**) J. E. Baldwin and C. H. Costain (in preparation).

A COVARIANT FORMULATION OF THE LAW OF CREATION OF MATTER

F. Hoyle

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Summary

A covariant law is given for the creation field of matter. When the electromagnetic and nuclear fields, and the stress within matter, are omitted from the energy-momentum tensor, the Einstein equations possess an infinity of isotropic, homogeneous, stable, steady-state solutions. The possibility exists that, if the neglected terms were reinstated, the solutions would possess a slow secular drift.

1. *Introduction.*—The object of the present paper is to formulate the steady-state model of the universe in a satisfactory covariant way. A brief review will first be given of the existing state of affairs, so far as theories based on the Einstein field equations are concerned. For considerations lying outside this framework the reader should consult Bondi and Gold (**1**).

Isotropic, homogeneous models of the universe possess a line-element of the form

$$ds^2 = c^2\,dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right], \tag{1}$$

where k can be zero or ± 1. The well-known first-order approximation to the red-shift magnitude relation follows already from (1), and therefore does not depend at all on the system of dynamics used to determine the precise variation of R with time t.

The dynamics of cosmological theory are determined by Einstein's equations,

$$G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G + \lambda g_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{2}$$

where $G_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is the energy-momentum tensor, $\kappa = 8\pi\gamma/c^4$ with γ the Newtonian constant, and λ is the cosmological constant. For a single-stream fluid with isotropic pressure, $T^{\mu\nu}$ contains the term

$$(p+\rho)v^\mu v^\nu - pg^{\mu\nu}, \tag{3}$$

where v^μ is the velocity dx^μ/ds, p is the material pressure, and ρ is the density measured by an observer moving with the fluid. (Even for this observer, the particles of the fluid are in random motion, so that the apparent masses of the particles differ from the rest masses. This difference must be included in ρ.)

Expression (3) does not comprise the whole energy-momentum tensor. Extra terms containing the electromagnetic field and the nuclear field must also be included. But these are usually taken to be negligible, and (1), (2), together with (3) for $T^{\mu\nu}$, define the well-known forms of relativistic cosmology described very clearly and in detail by Robertson (**2**).

A theory containing the creation of matter can be obtained (**3**) either by introducing a suitable term into (2) or by the entirely equivalent procedure of adding an extra term to $T^{\mu\nu}$. As was pointed out by McCrea (**4**) the second way of writing the theory is preferable because it emphasises the crucial point that the logical structure of the Einstein equations is not thereby changed.

Equation (2) connects a geometrical tensor of defined form on the left with an undefined physical tensor on the right—the structure of the relativity theory does not itself decide what form the physical tensor shall take. (In fact the formulation of $T^{\mu\nu}$ has changed since Einstein's paper of 1915, in the sense that the nuclear field must now be included in $T^{\mu\nu}$.) In particular, conservation equations, $T^{\mu\nu}{}_{,\nu}=0$, hold whatever the formulation of $T^{\mu\nu}$.

For a universe of infinite volume (as in the steady-state theory) energy conservation for the whole universe is a meaningless notion. Conservation of energy must be considered in relation to a box of finite volume. Two important cases evidently arise: the box can have a fixed proper volume, or its walls can expand as the universe expands. When there is creation of matter, the conservation equations require energy to be constant in a box of fixed proper volume. When there is no creation of matter, the conservation equations require energy to be constant in a box that expands with the universe.

To end these introductory remarks, it may be noted that if the time-axis is open to $-\infty$, as it is in the steady-state theory, the cases $k=0, \pm 1$ are indistinguishable over any finite portion of the universe. Hence for any comparison with observation it is sufficient to consider the case where k in (1) is taken as zero. The theory (**3**) was further simplified by putting $\lambda=0$. These simplifications will be retained in the following work.

2. *The creation term.*—The success of the isotropic, homogeneous theory in predicting the linear red-shift magnitude relation is a strong indication that the form (1) of the line element is correct (apart of course from detailed fluctuations due to local irregularities in the distribution of matter). Moreover, observation provides direct evidence that the large scale distribution of galaxies is in fact isotropic and homogeneous.

This raises the remarkable point that apparently a preferred direction is defined at every point of space-time, namely the direction of the universal time t, viz. (1, 0, 0, 0) in the system of coordinates used in (1). By differentiating this direction covariantly, a tensor $C_{\mu\nu}$ is obtained, of which

$$C_{11}=C_{22}=C_{33}=-R\dot{R} \tag{4}$$

are the only non-vanishing components. In essence, the procedure adopted in (**3**) was to multiply $C_{\mu\nu}$ by a constant α and to add $\alpha C^{\mu\nu}$ to (3) in order to form $T^{\mu\nu}$. The resulting dynamical equations have a steady-state solution

$$\begin{aligned} R(t) &= \exp(ct/a), \\ \rho &= 3c^2/8\pi\gamma a^2, \\ a &= 3c^3/8\pi\alpha\gamma. \end{aligned} \tag{5}$$

It was also shown that any fluctuation of $R(t)$ from this solution is exponentially damped in a time interval of order a/c.

In this way a unit of distance a, a mass density ρ, and a time unit a/c, were obtained. The emergence of universal units of length, mass, and time arises from the introduction of the creation constant α, which together with c, γ determine a and ρ. The time unit a/c is the reciprocal of the Hubble constant H.

The objection to this theory—that the definition of $C_{\mu\nu}$ makes appeal to a particular coordinate system—was answered (5) with the argument that no theory working entirely in terms of covariant laws (i.e. without reference to a special coordinate system) could explain the development of preferred directions in space-time. Underlying this argument was the concept that the ultimate aim of cosmological theory should be to establish the following result: given any arbitrary distribution of matter and motions, and arbitrary values of the metric tensor and its derivatives, on a space-like surface, prove that the universe must ultimately evolve to its present isotropic, homogeneous state. It is difficult to see how this result could ever follow from covariant laws, since neither the laws nor the initial situation single out any particular coordinate system for especial preference.

The point is so important as to be worth restating in a somewhat different way. One learns from the relativity theory that space and time are no more than a fourfold set of parameters by which physical events are ordered and that the parameters can be chosen in an infinity of equivalent ways. Experience in microscopic physics provides overwhelming evidence in favour of this point of view. Yet special coordinate systems arise in the macroscopic world, not only as a matter of convenience, but in a quite fundamental way. Newtonian theory is dependent on the concept of absolute space. At a more sophisticated level, Mach avoided the necessity for absolute space by choosing a coordinate system that was non-rotating and unaccelerated with respect to the distant bodies of the universe. At a still more sophisticated level, in the theory of relativity the behaviour of an isolated singularity in accordance with the Schwarzschild solution is dependent on the existence of a suitable isotropic boundary condition at large distance from the singularity. It is of course the isotropic character of the universe that supplies this boundary condition. Indeed it is in just this way that the special coordinate system determined by the large scale structure of the universe impresses itself on everyday experience—for example, in controlling the development of cyclones in the atmosphere of the Earth.

The dilemma that exists between macroscopic experience on the one hand and the covariant outlook of microscopic physics on the other hand is therefore very real and serious. The dilemma is side-stepped by an unsatisfactory device in systems of cosmology where the negative time-axis is closed. In such cosmologies the universe is supposed to have been created at a finite value of t. In the mathematical sense, 'creation' means that certain boundary conditions are imposed on the differential equations of physics. Isotropy and homogeneity simply make their appearance at the outset among these boundary conditions.

When the time-axis is open to $-\infty$ the question cannot be dismissed so easily—or so artificially. Two points of view can be put forward. One of these has already been mentioned—of starting with an entirely arbitrary distribution of matter and of proving the development of isotropy and homogeneity. This point of view apparently demands the adoption of some non-covariant law.

Alternatively, one can argue that such a demonstration would be quite unnecessarily strong, since all that really needs proving is the *stability* of a large scale isotropic homogeneous distribution. According to this second point of view, the notion of starting with an arbitrary distribution of matter could well be an invalid concept, because the universe need never have been in any state other than one of homogeneity and isotropy however far back in time we go.

Although the choice of procedure is to some extent a question of philosophy, the possibility that the law of creation can be made entirely covariant in the second case is evidently of physical significance. The problem therefore arises in the following form: how shall a covariant law of creation be formulated that maintains permanent large-scale homogeneity and isotropy? In the following section a law will be obtained that gives timewise stability. Spacewise stability is not strictly proven, but the very form of the law makes spatial stability extremely probable.

3. *A covariant creation law.*—It is proposed that matter be thought of as radiating a creation field. For simplicity, the field will be taken scalar, thus

$$\Box\phi = K\rho, \tag{6}$$

where K is a constant, ρ is the mass density, and $\Box$ implies that the field φ is differentiated twice covariantly and that the two suffices are then contracted. This field presumably arises, if it exists at all, from the microscopic processes of fundamental physics. Although at present nothing is known of what these processes might be, it is reasonable to suppose that the source of the field would be proportional to the mass density ρ. Uncertainty concerning the microscopic origin of the field is therefore limited to uncertainty in the value of K. (The present formalism includes the case where there is no field φ, this being given by $K = 0$.)

Differentiation of φ gives a preferred direction at every point where the gradient is non-vanishing. A preferred direction could also have been obtained from a vector field, but there would be a far greater uncertainty in the nature of the source function in the case of a vector field. Thus for a field φ_μ the equation corresponding to (6) would be

$$\Box\varphi_\mu = K_\mu\rho,$$

and a considerable knowledge of the microscopic processes would be needed to determine the direction of K_μ at each source particle.

To obtain a system of cosmological dynamics, $C_{\mu\nu}$ is first obtained by differentiating φ twice, viz.

$$C_{\mu\nu} = \frac{\partial^2\phi}{\partial x^\mu \partial x^\nu} - \Gamma_{\mu\nu}^{\lambda}\frac{\partial\phi}{\partial x^\lambda}, \tag{7}$$

where $\Gamma_{\mu\nu}^{\lambda}$ is the usual Christoffel symbol. The tensor $C_{\mu\nu}$ is then added to (3) in order to form $T_{\mu\nu}$. (On this occasion nothing is gained by multiplying by a constant before $C_{\mu\nu}$ is added to $T_{\mu\nu}$, since any such constant can be absorbed into ϕ.)

It is seen from (7) that $C_{\mu\nu}$ is of necessity symmetric in its two suffices, so that the symmetry of $T_{\mu\nu}$ is not impaired by this procedure.

Consider now the dynamical equations for the smoothed case in which φ is a function of t only. With $k = 0$, the line element (1) can be written in terms of coordinates t, x_1, x_2, x_3, very simply as

$$ds^2 = c^2\,dt^2 - R^2(t)\,(dx_1{}^2 + dx_2{}^2 + dx_3{}^2). \tag{8}$$

The only non-vanishing Christoffel symbols are

$$\Gamma_{ij}{}^0 = R\dot{R}\,\delta_{ij}/c^2, \quad \Gamma_{0j}{}^i = \dot{R}\delta_{ij}/R; \quad i, j = 1,\ 2,\ 3, \tag{9}$$

while the only non-vanishing components of $G_{\mu\nu}$ are

$$\begin{aligned} G_{ij} &= -(R\ddot{R} + 2\dot{R}^2)\,\delta_{ij}/c^2; \qquad i, j = 1,\ 2,\ 3, \\ G_{00} &= 3\ddot{R}/R. \end{aligned} \tag{10}$$

From (10) it follows easily that

$$G = 6(R\ddot{R} + \dot{R}^2)/R^2c^2. \tag{11}$$

The non-vanishing components of $C_{\mu\nu}$ are

$$C_{ij} = -R\dot{R}\dot{\varphi}\,\delta_{ij}/c^2; \qquad i,j=1,\ 2,\ 3, \tag{12}$$
$$C_{00} = \ddot{\varphi}.$$

Three independent dynamical equations follow from (2), (7), and from the construction of $T_{\mu\nu}$. With $p=0$ in (3) and taking $v^{\mu}=c(1, 0, 0, 0)$, these equations are

$$2R\ddot{R} + \dot{R}^2 - \kappa R\dot{R}\dot{\phi} = 0, \tag{13}$$

$$3\frac{\dot{R}^2}{R^2} = \kappa(\ddot{\varphi} + \rho c^4), \tag{14}$$

$$3\dot{\varphi}\frac{\dot{R}}{R} = Kc^2\rho - \ddot{\phi}\,. \tag{15}$$

Consider now the condition for a steady-state solution, $R=\exp(Ht)$ with H a constant. From (13) we require

$$\dot{\varphi} = 3H/\kappa, \quad \text{i.e. } \varphi = 3Ht/\kappa, \quad \kappa = 8\pi\gamma/c^4. \tag{16}$$

Thus $\ddot{\varphi}=0$ and (14) gives

$$\rho = 3H^2/8\pi\gamma, \tag{17}$$

as in the previous theory (3). Substituting for $\dot{\varphi}$, ρ in (15) now gives $K=3c^2$. Thus unless (6) takes the form

$$\Box\phi = 3c^2\rho, \tag{18}$$

a steady-state solution cannot be found at all.

This result is most satisfactory since it removes the former arbitrariness of a multiplying constant (the constant α in section 2). Thus the steady-state theory determines the precise law of the creation field, viz (18), thereby making a definite statement as to the form of the law that microscopic physics should reveal.

The disappearance of the multiplying constant α means that universal units of length, mass, and time are no longer defined, however. The difference shows itself in the fact that, provided φ satisfies (18), a steady-state solution of (13), (14), and (15) can be found for any value of the Hubble constant H. A discussion of whether this is an advantage or not will be deferred until after the timewise stability of the equation has been discussed.

It may be observed that the form of (18) requires φ to be determined always by the distant matter of the universe, so that φ is not appreciably affected by local fluctuations in the distribution of matter. For this reason it is to be expected that the universe is spacewise stable.

4. *Stability with respect to time.*—Suppose (18) is satisfied. Consider a perturbation from the steady-state solution $R=\exp(H_0 t)$, where H_0 is a constant. The perturbation can be expressed by

$$\frac{\dot{R}}{R} = H_0(1+\xi), \quad \xi \ll 1. \tag{19}$$

Differentiating (19) gives

$$\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} = H_0\dot{\xi}\,. \tag{20}$$

By substitution for $\dot{R}$, $\ddot{R}$ in (13) we then have

$$\kappa\dot{\phi} = 3H_0(1+\xi) + 2\dot{\xi}, \tag{21}$$

a second order term in 3ξ being neglected. Putting $K=3c^2$ in (15), and eliminating ρ between (14) and (15), gives

$$9\frac{\dot{R}^2}{R^2} = 3\kappa\dot{\phi}\,\frac{\dot{R}}{R} + 4\kappa\ddot{\phi}. \tag{22}$$

By differentiating (21) to give

$$\kappa\ddot{\varphi} = 3H_0\dot{\xi} + 2\ddot{\xi}. \tag{23}$$

and by substitution for $\dot{R}/R$, φ, $\ddot{\varphi}$ in (22), we finally obtain

$$4\dot{\xi} + 9H_0\xi = 0,$$

with a general solution of the form

$$\xi = A\exp(-9H_0t/4) + B. \tag{24}$$

Hence we see that a small perturbation of the universe introduces a damped term, together with a constant shift of the Hubble constant. The possibility of such a shift is obviously to be expected, since a steady-state solution can be found at any value of H. The interest therefore lies in the stability of the time dependent term. The universe evidently has no exponentially unstable solution. Nor is there any solution in small oscillations.

5. *Discussion.*—The point of view of the present theory is that the energy momentum tensor $T_{\mu\nu}$ contains four parts, viz.

$$T^{\mu\nu} = T_g^{\ \mu\nu} + T_e^{\ \mu\nu} + T_c^{\ \mu\nu} + T_n^{\ \mu\nu}, \tag{25}$$

and that (25), together with the Einstein relation (1), determines the dynamical equations of cosmology. The quantity $T_g^{\ \mu\nu}$, given by (3) for a single stream fluid with isotropic pressure, generates the ordinary gravitational field. The electromagnetic energy-momentum tensor $T_e^{\ \mu\nu}$ is given by

$$T_e^{\ \mu\nu} = 1/4\pi\left(-F^{\mu\nu}F_\lambda^{\ \nu} + \tfrac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right), \tag{26}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. The energy momentum tensor of the creation field is given by

$$\Box\phi = 3c^2\rho,$$
$$T_c^{\ \mu\nu} = \varphi^{\mu,\nu}, \tag{27}$$

the covariant form of $\phi^{\mu,\nu}$ being written out explicitly in (7). No canonical form for the nuclear energy-momentum tensor $T_n^{\ \mu\nu}$ is at present known.

In this way the creation field plays an entirely similar role to the other fields. From a cosmological point of view the important terms in (25) are $T_g^{\ \mu\nu}$, $T_c^{\ \mu\nu}$. Ironically, it is $T_e^{\ \mu\nu}$, $T_n^{\ \mu\nu}$ that are studied so extensively in microscopic physics.

It is possible to take two views of the present value of the expansion rate $\dot{R}/R$. Without perturbations, and accepting (27), the universe is steady, in which case there need never have been any other value of the expansion rate. It is then meaningless to attempt to find a reason for the observed value, $\dot{R}/R = 2{\cdot}5 \times 10^{-18}\,\text{sec}^{-1}$.

On the other hand it is possible that perturbations, satisfying (24) with $B \neq 0$, arise from $T_e^{\ \mu\nu}$, $T_n^{\ \mu\nu}$, or simply from local spatial fluctuations. In such a case the universe would possess a slow secular drift in its expansion rate H. Consequently the mean density of matter, given by (17), would also possess a secular drift.

Whether or not such a drift would actually occur is excessively difficult to determine. The main contribution to $T_e^{\ \mu\nu}$ comes from starlight. The main contribution to $T_n^{\ \mu\nu}$ presumably comes from nuclear binding and from the neutrino field (these have been lumped together into one term). It should be a good approximation to regard these tensors as spatially smooth, in which case it is to be expected that the constant K in (6) can be chosen so as to permit a stable, steady solution $R = \exp(Ht)$ of the dynamical equations, *provided the tensors $T_e^{\ \mu\nu}$, $T_n^{\ \mu\nu}$ are steady*. This is the doubtful point, however. The formation of galaxies, and the incidence of nuclear activity inside stars, must be affected by the value of H.

Hence a secular drift of H must change $T_e^{\mu\nu}$, $T_n^{\mu\nu}$, which in turn produces feed-back into the dynamical equations themselves. If the feed-back were positive, drift would continue.

In so far as any criterion can be given for judging a system of cosmological dynamics—other than the overriding observational criterion—it might be this: that the ' best ' system of cosmology is the one that admits of the greatest variety of physical possibilities. If this criterion is admitted, a universe with slow drift is to be preferred to an entirely steady-state universe at fixed H. In the latter case only one unique set of properties can be worked through, namely those that correspond to the one unique value of H. In the case of secular drift, on the other hand, an infinity of different sets of properties can be examined; for if the drift is slow enough the full physical implication of each value of H can be established in turn. (Since different H values imply different values of the mean density ρ, any marked shift of H certainly implies a non-trivial change in the properties of the universe.) Defining H^{-1} as the length of a ' generation ', with H always taking its instantaneous value, it is possible if the drift were sufficiently slow to proceed backwards along the time-axis through an infinity of generations and yet for the universe never exactly to have repeated itself in any two of these generations.

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References

(1) Bondi, H., and Gold, T., *M.N.*, **108**, 252, 1948.
(2) Robertson, H. P., *Rev. Mod. Phys.*, **5**, 62, 1933.
(3) Hoyle, F., *M.N.*, **108**, 372, 1948.
(4) McCrea, W. H., *Proc. Roy. Soc.* A, **206**, 562, 1951.
(5) Hoyle, F., *M.N.*, **109**, 365, 1949.

THE HUND GRAVITATIONAL EQUATIONS AND THE EXPANDING UNIVERSE

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Summary

It is shown that the Hund gravitational equations can be applied to a spherically expanding universe. The experimentally determined constants are used to measure the age of the universe.

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1. *Introduction.*—A variety of cosmologies have been invented to describe the nature of the universe. Any cosmology must, in order to be considered as representing the universe, be consistent with two macroscopic properties of the physical universe. First, the mean density of the universe is constant over space within measurable distances and, secondly, the velocity of recession of nebulae (the red shift) is proportional to the first approximation to the distance from the observer. The object of this paper is to provide a semi-Newtonian cosmology for the universe, and from this to determine the age.

Hund (1) has suggested that gravitational fields may be expressed in terms of a polar vector $\mathbf{f}$ and an axial vector $\mathbf{g}$ which act upon any gravitational mass m with the force

$$m\{\mathbf{f}+(\mathbf{v}/c)\times\mathbf{g}\} \tag{1.1}$$

$\mathbf{v}$ being the velocity of the mass. The fields $\mathbf{f}$ and $\mathbf{g}$ satisfy the following differential equations.

$$\nabla\cdot\mathbf{g}=0 \tag{1.2}$$

$$\nabla\times\mathbf{f}=-\frac{1}{c}\frac{\partial\mathbf{g}}{\partial t} \tag{1.3}$$

$$\nabla\cdot\mathbf{f}=-4\pi\gamma\rho+(f^2+g^2)/(2c^2) \tag{1.4}$$

$$\nabla\times\mathbf{g}=\frac{1}{c}\frac{\partial\mathbf{f}}{\partial t}-\frac{4\pi\gamma\rho\mathbf{v}}{c}+\frac{1}{c^2}(\mathbf{f}\times\mathbf{g}) \tag{1.5}$$

ρ is the mass density.

The fields $\mathbf{f}$ and $\mathbf{g}$ are supposed to be produced by gravitating masses and currents, analogous to the electromagnetic field. All forms of energy and energy flux, including the fields due to $\mathbf{f}$ and $\mathbf{g}$ themselves, are considered as sources of gravitation and so those terms in the field equations which correspond to the source terms in Maxwell's equations must include in addition terms involving $\mathbf{f}$ and $\mathbf{g}$.

The rate of creation of mass per unit volume is derived from these equations to be

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=\frac{1}{c^2}\rho\mathbf{f}\cdot\mathbf{v} \tag{1.6}$$

and this is the rate at which gravitation does work on the system divided by c^2.

These equations may be used to provide a smoothed out representation of the universe. If it be assumed that everything is radially symmetric **g** vanishes and **f** and **v** are in the radial direction.

Under these conditions equations (1.2)–(1.5) reduce to

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f) = -4\pi\gamma\rho + f^2/(2c^2) \tag{1.7}$$

$$\frac{\partial f}{\partial t} = 4\pi\gamma\rho v. \tag{1.8}$$

Equation (1.6) becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v) = \frac{1}{c^2} f v \rho. \tag{1.9}$$

Furthermore, if it be assumed that the motion of the universe is generated solely by its own gravitational interactions, f will be the acceleration of the matter under consideration and will be related to v by

$$f = \frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} \tag{1.10}$$

where $\frac{d}{dt}$ denotes differentiation following the motion.

Equations (1.7), (1.8) and (1.10) are three differential equations relating the three quantities f, v, ρ.

Having solved these equations, the quantity $(fv\rho)/c^2$, the rate of creation of mass per unit volume may be derived. It follows that in this theory there is bound to be creation as $(fv\rho)/c^2$ cannot vanish except for the trivial zero solutions of equations (1.7) (1.8) and (1.10). This, as mentioned previously, is equivalent to the rate at which gravitation does work. The actual solution of equations (1.7) (1.8) and (1.10) is clearly an almost impossible task. It is however possible to find approximate solutions which hold within certain ranges.

2. *Application to a class of model universes.*—If ρ is uniform, Hund (**1**) showed that the solution of (1.7) which is finite at $r=0$, is

$$f = 2c^2\left\{\frac{1}{r} - \frac{\sqrt{2\pi\gamma\rho}}{c}\coth\frac{\sqrt{2\pi\gamma\rho}\, r}{c}\right\}. \tag{2.1}$$

Now if $(\sqrt{2\pi\gamma\rho}\, r)/c$ is small

$$f = -\tfrac{4}{3}\pi\gamma\rho r. \tag{2.2}$$

This expression will be used to determine an expression for the time variation of density. This will be legitimate provided $(\sqrt{2\pi\gamma\rho}\, r)/c$ is small.

Using equation (1.8)

$$v = -\frac{1}{3\rho}\frac{d\rho}{dt} = r/\tau \quad \text{at present} \tag{2.3}$$

where τ is the reciprocal of Hubble's constant (**2**). It will be seen that v is proportional to r. Furthermore because there is recession v is positive and $\frac{d\rho}{dt}$ is negative.

Substituting equations (2.2) and (2.3) in equation (1.10)

$$-\tfrac{4}{3}\pi\gamma\rho r = -\tfrac{1}{3}r\left\{\frac{1}{\rho}\frac{d^2\rho}{dt^2} - \frac{1}{\rho^2}\left(\frac{d\rho}{dt}\right)^2\right\} + \left\{-\tfrac{1}{3}r\frac{1}{\rho}\frac{d\rho}{dt}\right\}\left\{-\frac{1}{3\rho}\frac{d\rho}{dt}\right\}.$$

This simplifies to

$$\frac{1}{\rho}\frac{d^2\rho}{dt^2} - \frac{4}{3\rho^2}\left(\frac{d\rho}{dt}\right)^2 = 4\pi\gamma\rho. \tag{2.4}$$

This may be rewritten as

$$\frac{d}{dt}\left\{\rho^{-8/3}\left(\frac{d\rho}{dt}\right)^2\right\} = 8\pi\gamma\rho^{-2/3}. \tag{2.5}$$

This integrates to

$$\left(\frac{d\rho}{dt}\right)^2 = 24\pi\gamma\rho^3 + K\rho^{8/3} \tag{2.6}$$

If ρ_0 is the present density, it follows that

$$\left(\frac{1}{3\rho}\frac{d\rho}{dt}\right)^2 = \frac{8}{3}\pi\gamma\rho_0\left\{\frac{\rho}{\rho_0} + \left(\frac{3}{8\pi\gamma\rho_0\tau^2} - 1\right)\left(\frac{\rho}{\rho_0}\right)^{2/3}\right\} \tag{2.7}$$

and the nature of the solution depends on the sign of

$$3/(8\pi\gamma\rho_0\tau^2) - 1 = \Delta \text{ say.}$$

Writing $(\rho/\rho_0)^{1/3} = x$, equation (2.7) becomes

$$\left(\frac{1}{x}\frac{dx}{dt}\right)^2 = \frac{8\pi\gamma\rho_0}{3}\{x^3 + \Delta x^2\}. \tag{2.8}$$

The behaviour of x is qualitatively similar to that of ρ.

(*a*) If Δ is zero the solution of equation (2.4) is

$$\rho = \frac{1}{6\pi\gamma(t+t_0)^2} = \frac{\rho_0 t_0^2}{(t+t_0)^2} \tag{2.9}$$

if $t = 0$ be the present epoch.

This implies that the density was originally infinite at a time $t_0 = (6\pi\gamma\rho_0)^{-1/2}$ back and will decrease steadily becoming zero at infinite time. t_0 is thus the age of this model universe.

(*b*) If Δ is positive ($= a$ say) equation (2.8) can be rewritten as

$$\left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t = \int_x^1 \frac{dy}{y^2(y+a)^{1/2}} \tag{2.10}$$

$$= -\frac{1}{2a^{3/2}}\left\{\log\left(\frac{\sqrt{a+x}-\sqrt{a}}{\sqrt{a+x}+\sqrt{a}}\right) - \log\left(\frac{\sqrt{a+1}-\sqrt{a}}{\sqrt{a+1}+\sqrt{a}}\right)\right\}$$

$$+\frac{\sqrt{a+x}}{ax} - \frac{\sqrt{a+1}}{a}. \tag{2.11}$$

Here again x (and hence ρ) decreases steadily with time.

At time t_0 past, where

$$\left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t_0 = \frac{\sqrt{a+1}}{a} - \frac{1}{2a^{3/2}}\log\left(\frac{\sqrt{a+1}-\sqrt{a}}{\sqrt{a+1}-\sqrt{a}}\right), \tag{2.12}$$

x (and ρ) were infinite. If a tends to infinity, t_0 tends to τ. t_0 is thus the age of this model universe. At infinite time in the future x (and ρ) will be zero. τ is thus an upper bound to the age of the universe.

(*c*) If Δ is negative ($\Delta = -b$ say).

$$\left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t = \int_x^1 \frac{dy}{y^2(y-b)^{1/2}} \quad (y \geqslant x \geqslant b). \tag{2.13}$$

The situation here is slightly different. x (and ρ) decrease steadily with time until $x = b$ when $t = t'$. After that time x (and ρ) will increase again until x becomes infinite. This takes a further time $t' + t_0$ where t_0 is the time past at which x was originally infinite and is thus the age of this model universe.

$$\left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t_0 = \int_1^\infty \frac{dy}{y^2(y-b)^{1/2}}$$
$$= \frac{1}{b^{3/2}}\left\{\frac{\pi}{2} - \cos^{-1} b^{1/2}\right\} - \frac{(1-b)^{1/2}}{b}. \tag{2.14}$$

The time t' is given by

$$\left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t' = \int_b^1 \frac{dy}{y^2(y-b)^{1/2}}$$
$$= \frac{(1-b)^{1/2}}{b} + \frac{1}{b^{3/2}} \cos^{-1} b^{1/2}$$
$$= \frac{\pi}{2b^{3/2}} - \left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t_0. \tag{2.15}$$

Hence
$$2(t_0 + t') = \frac{\pi}{b^{3/2}}\left(\frac{3}{8\pi\gamma\rho_0}\right)^{1/2} \tag{2.16}$$

is thus the lifetime of the universe from its inception at infinite density, through its expansion to a density $\rho_0 b^3$, to its cessation at infinite density again.

It may be remarked that this analysis holds only provided that $(\sqrt{2\pi\gamma\rho}\, r/c)$ is small, and as ρ becomes infinite, the corresponding value of r becomes zero. Thus the solution is not strictly true at the limits $-t_0$ and t', but will give an idea of the manner in which these limits vary with the present density of the universe.

3. *Application of observational values.*—The presently accepted value of Hubble's constant is given by (**2**).

$$\tau^{-1} = 2\cdot5 \times 10^{-18}\,\text{sec}^{-1}, \quad \tau = 13 \times 10^9\,\text{years} \tag{3.1}$$

and it is estimated that the present mean density of the universe is between

$$10^{-25} \text{ and } 10^{-31}\,\text{gm/cc } (\mathbf{3}). \tag{3.2}$$

Now if the present mean density is $1\cdot08 \times 10^{-29}$ gm/cc

$$3/(8\pi\gamma\rho_0\tau^2) = 1 \tag{3.3}$$

and Δ vanishes, and so in order to discuss the effect of mean density, the following values of ρ_0 will be taken, which give a fair spread.

Table I

	ρ_0 (gm/cc)	$3/(8\pi\gamma\rho_0\tau^2)$	Δ	
(*a*)	$1\cdot08 \times 10^{-30}$	10	9	$a = 9$.
(*b*)	$1\cdot08 \times 10^{-29}$	1	0	
(*c*)	$1\cdot08 \times 10^{-28}$	$\cdot1$	$-0\cdot9$	$b = 0\cdot9$
(*d*)	$1\cdot08 \times 10^{-25}$	10^{-4}	$-0\cdot9999$	$b = 0\cdot9999$.

Case (a): $\rho_0 = 1{\cdot}08 \times 10^{-30}$ gm/cc

$$\left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t_0 = \frac{\sqrt{10}}{9} - \frac{1}{27}\log(\sqrt{10}+3), \tag{3.4}$$

which may be rewritten as

$$t_0/\tau = \frac{10}{9} - \frac{\sqrt{10}}{27}\log(\sqrt{10}+3) \tag{3.5}$$

whence

$$t_0 = 0{\cdot}90\tau \tag{3.6}$$

$$= 11{\cdot}7 \times 10^9 \text{ years.} \tag{3.7}$$

Case (b): $\rho_0 = 1{\cdot}08 \times 10^{-29}$ gm/cc

$$t_0 = 1/(6\pi\gamma\rho_0)^{1/2} \tag{3.8}$$

$$= \tfrac{2}{3}\tau \tag{3.9}$$

$$= 8{\cdot}7 \times 10^9 \text{ years.} \tag{3.10}$$

Case (c): $\rho_0 = 1{\cdot}08 \times 10^{-28}$ gm/cc

$$\left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t_0 = \frac{\pi}{2b^{3/2}} - \left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t' \tag{3.11}$$

where

$$\left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t' = \frac{(1-b)^{1/2}}{b} + \frac{1}{b^{3/2}}\sin^{-1}(1-b)^{1/2}. \tag{3.12}$$

Now it will be seen that as we are only interested in orders of magnitude we may assume $(1-b)^{1/2}$ small. (If ρ_0 be one degree of magnitude smaller, the formulae (2.13–2.16) do not apply, whereas if ρ_0 be greater the approximation is *a fortiori* better.)

Taking then terms to the first order only in $(1-b)^{1/2}$, (3.12) becomes

$$\left(\frac{8\pi\gamma\rho_0}{3}\right)^{1/2} t' = 2(1-b)^{1/2} \tag{3.13}$$

$$= 2\left(\frac{3}{8\pi\gamma\rho_0\tau_2}\right)^{1/2} \tag{3.14}$$

and

$$t'/\tau = 3/(4\pi\gamma\rho_0\tau^2) \tag{3.15}$$

and from (3.11)

$$\frac{t_0}{\tau} = \frac{\pi}{2}\left(\frac{3}{8\pi\gamma\rho_0\tau_2}\right)^{1/2} - \frac{t'}{\tau}. \tag{3.16}$$

For the value of ρ_0 under consideration $3/(8\pi\gamma\rho_0\tau^2) = 0{\cdot}1$ and so

$$t' = 0{\cdot}20\tau \tag{3.17}$$

$$= 2{\cdot}6 \times 10^9 \text{ years} \tag{3.18}$$

$$t_0 = 0{\cdot}30\tau \tag{3.19}$$

$$= 3{\cdot}9 \times 10^9 \text{ years.} \tag{3.20}$$

Case (*d*): $\rho_0 = 1{\cdot}08 \times 10^{-25}$ gm/cc.

In this case

$$3/(8\pi\gamma\rho_0\tau^2) = 10^{-4} \tag{3.21}$$

$$t' = 2 \times 10^{-4}\tau \tag{3.22}$$

$$= 2{\cdot}6 \times 10^6 \text{ years.} \tag{3.23}$$

This high density appears to predict a very short future for the universe.

$$\frac{t_0}{\tau} = \frac{\pi}{2}(0{\cdot}01) - 2 \times 10^{-4} \tag{3.24}$$

$$= 0{\cdot}0155$$

$$t_0 = 0{\cdot}20 \times 10^9 \text{ years.} \tag{3.25}$$

Here again the time scale is on the small side. The age of the universe for various present mean densities is given by the following table

ρ_0 (gm/cc)	(t_0/τ)	t_0 (years)
0	1	13×10^9
$1{\cdot}08 \times 10^{-30}$	0·90	$11{\cdot}7 \times 10^9$
$1{\cdot}08 \times 10^{-29}$	0·67	$8{\cdot}7 \times 10^9$
$1{\cdot}08 \times 10^{-28}$	0·30	$3{\cdot}9 \times 10^9$
$1{\cdot}08 \times 10^{-25}$	0·0155	$0{\cdot}2 \times 10^9$

The range of applicability of these calculations is given by $(\sqrt{2\pi\gamma\rho_0}\, r)/c$ small. Taking the value $1{\cdot}08 \times 10^{-29}$ gm/cc for ρ_0,

$$2\pi\gamma\rho_0 = 3/(4\tau^2) \tag{3.26}$$

implying for applicability that

$$(r\sqrt{3})/(2c\tau) \ll 1 \tag{3.27}$$

i.e. that

$$r \ll \frac{2}{\sqrt{3}} c\tau \tag{3.28}$$

$$= 15 \times 10^9 \text{ light years.} \tag{3.29}$$

This distance is considerably greater than the distance of observed nebulae (**4**) and so the approximations involved in this paper are valid.

4. *Suggested further work.*—If

$$t = T/(4\pi\gamma\rho_0)^{1/2} \tag{4.1}$$

$$r = cR/(4\pi\gamma\rho_0)^{1/2} \tag{4.2}$$

$$\rho = \rho_0 P \tag{4.3}$$

$$v = cV \tag{4.4}$$

$$f = c(4\pi\gamma\rho_0)^{1/2} F \tag{4.5}$$

equations (1.7) (1.8) (1.10) can be expressed non-dimensionally as

$$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 F) = -P + \tfrac{1}{2}F^2 \tag{4.6}$$

$$\frac{\partial F}{\partial T} = PV \tag{4.7}$$

$$F = \frac{\partial V}{\partial T} + V\frac{\partial V}{\partial R}. \tag{4.8}$$

These form a set of three partial differential equations in three variables, and are amenable to numerical analysis techniques on the modern large scale computers.

Basically the interest is in P and V, F being an auxiliary, and it should be possible to solve equations (4.6)–(4.8) to provide various model universes.

Power series solutions of equations (4.6)–(4.8) may be obtained as follows:— Let

$$P = \sum_{n=0}^{\infty} P_n(T) R^{2n} \tag{4.9}$$

$$F = \sum_{n=0}^{\infty} F_n(T) R^{2n+1} \tag{4.10}$$

$$V = \sum_{n=0}^{\infty} V_n(T) R^{2n+1}. \tag{4.11}$$

Equations (4.6)–(4.8) become respectively:

$$\sum_{n=0}^{\infty} (2n+3) F_n R^{2n} + \sum_{n=0}^{\infty} P_n R^{2n} = \tfrac{1}{2} \sum_{n=1}^{\infty} \sum_{s=1}^{n=1} F_s F_{n-s-1} R^{2n} \tag{4.12}$$

$$\sum_{n=0}^{\infty} F_n' R^{2n+1} = \sum_{n=0}^{\infty} \sum_{s=0}^{n} P_{n-s} V_s R^{2n+1} \tag{4.13}$$

$$\sum_{n=0}^{\infty} F_n R^{2n+1} = \sum_{n=0}^{\infty} V_n' R^{2n+1} + \sum_{n=0}^{\infty} (n+1) R^{2n+1} \sum_{s=0}^{n} V_s V_{n-s}. \tag{4.14}$$

The dash denotes differentiation with respect to T. The leading terms give

$$3F_0 = -P_0 \tag{4.15}$$

$$F_0' = P_0 V_0 \tag{4.16}$$

$$F_0 = V_0' + V_0^2 \tag{4.17}$$

and the other terms may be rearranged to give

$$(2n+3) F_n + P_n = \tfrac{1}{2} \sum_{s=0}^{n-1} F_s F_{n-s-1} \tag{4.18}$$

$$F_n' - P_n V_0 - V_0 P_n = \sum_{s=1}^{n-1} P_{n-s} V_s \tag{4.19}$$

$$F_n - V_n' - 2(n+1) V_0 V_n = (n+1) \sum_{s=1}^{n-1} V_s V_{n-s}. \tag{4.20}$$

In equations (4.18)–(4.20), the coefficients of order n are expressed in terms of the coefficients of order $(n-1)$, and thus a solution can be obtained.

Equations (4.15)–(4.17) are the non-dimensional forms of the equations dealt with in section 2. One further point of interest arises from equations (4.9)–(4.11). The non-linearity of the recession velocity-distance relationship is usually expressed

$$v = kr + lr^2 + \ldots. \tag{4.21}$$

From equation (4.11), it would appear that a more suitable relationship is

$$v = kr + lr^3 + \ldots \tag{4.22}$$

and of course, for the integration of equations (4.18)–(4.20) these higher coefficients would have to be determined.

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References

(**1**) Hund, F., *Z. Physik.*, **124**, 742, 1948.
(**2**) Sandage, A., *Ap. J.*, **127**, 513, 1958.
(**3**) Proceedings of Solvay Conference, 1958.
(**4**) Humason, M. L., *Vistas in Astronomy*, Vol. II (Pergamon, 1958), 1620.

ANGULAR MEASUREMENTS IN OBSERVATIONAL COSMOLOGY

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Summary

Theoretical formulae are derived which, by expansion in series, relate the observed angular diameters and angular separations of galaxies, radio sources, and clusters in an expanding universe to either the measured red shift or registered apparent magnitude of representative sources in the region of survey.

It is shown that in principle the cosmological problem may be solved by fitting the observed data to these relations. In particular an evolutionary universe may be distinguished from a steady-state universe, and the Hubble parameter and acceleration parameter for either universe may be determined. Allowances are made for possible evolutionary trends in the sources, and it is shown how a progressive variation in intrinsic size may be detected.

Some exact relations between observables are provided for the steady-state model.

1. *Introduction*.—The apparent sizes of distant galaxies and of clusters of galaxies, as indicated by their suitably defined angular diameters, and also the angular separations of these objects, provide in principle very important criteria for the resolution of the cosmological problem, and in particular for distinguishing between an evolutionary and a steady-state universe. It is therefore very much to be hoped that statistical measurements of this nature will contribute reliable data towards the solution of the problem.

In this paper formulae are derived to provide an analytic basis for such data. In these relations the angular diameters and the angular separations of galaxies, clusters of galaxies, and radio sources, allowing for possible evolutionary trends, are connected suitably to the red shift exhibited by representative sources in the region of survey and also, alternatively, to the apparent magnitude of these sources.

As in the earlier papers in this series on observable relations in cosmology, our formulae are designed to connect observables that are, in principle, directly measured, and to take account of possible systematic changes with epoch. The previous papers, which we shall refer to as I (**1**) and II (**2**), dealt with the red shift—apparent magnitude relation for galaxies (I), and with the number counts of galaxies and radio sources (II). The present paper caters for an observational programme that may be regarded as either auxiliary to the programmes covered by these papers, or as an independent approach to the solution of the cosmological problem.

It should be mentioned that a valuable paper, dealing specifically with the ratio of the angular separation of neighbouring clusters of galaxies to their angular diameter as a function of red shift, has been published recently by Florides and McCrea (**3**). We shall have occasion to refer to this paper in our own work here. We attempt in this paper, however, to cover a wider programme of observational tests, including those that are independent of the requirement by Florides and

McCrea that the objects surveyed should not be gravitationally bound to one another. Also, it will be a feature of this paper that the intrinsic size of an object is regarded as in general a function of epoch, and we shall in principle be able to discover by our observable relations whether in fact a dependence on epoch exists. In addition, as well as distinguishing between an evolutionary and a steady-state universe, the relations found will allow us to identify certain characteristic parameters of the particular model that best fits the data. The test of Florides and McCrea does not distinguish between the various evolutionary models; this was, of course, one of the advantages claimed for their test when it was originally suggested by McCrea (**4**), namely that it is independent of a knowledge of the geometry of space time. However, if the universe is evolutionary a knowledge of its particular character will obviously be required, and the present paper goes some way in providing a means for its determination.

For the reasons stated in I and II we shall consider it adequate for our purpose to adopt the well-known Robertson–Walker space-time metric of an expanding universe, in which space is homogeneous and isotropic and a cosmic time exists, in the form

$$dS^2 = c^2\,dt^2 - \frac{R^2(t)(dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2)}{(1 + kr^2/4)^2}. \tag{1.1}$$

Here $k = 1$, 0, or -1 according as the uniform curvature of space (t = constant) is positive, zero, or negative respectively.

According to the usual interpretation of this ideal universe its material contents move on the geodesics r = const., θ = const., ϕ = const. In order to correspond with the observed phenomena, however, we shall suppose that, while the centres of clusters of galaxies move on these geodesics and thus take part in the expansion of the universe, the individual member galaxies will not necessarily do so, but may be gravitationally bound within the cluster. The r, θ, ϕ coordinates of such galaxies would in this case vary with the epoch t.

It will be convenient, as in I and II, to change the metric (1.1) to the form

$$dS^2 = c^2\,dt^2 - \frac{R^2(t)}{R_0^2}\left\{\frac{d\rho^2 + \rho^2\,d\theta^2 + \rho^2 \sin^2\theta\,d\phi^2}{(1 + \rho^2/a^2)^2}\right\} \tag{1.2}$$

by means of the substitution

$$\rho = R_0 r, \tag{1.3}$$

where $R_0 \equiv R(t_0)$ the value of the function $R(t)$ at the present epoch of observation t_0 (fixed). We see that the constant a^2 is equal to $4R_0^2/k$.

We repeat here the series expansions of relevant cosmological parameters which were established in I:

If t is the epoch of emission of the radiation from a distant source received at the epoch of observation t_0, we may write

$$R(t) = R_0(1 - \alpha_1\tau + \tfrac{1}{2}\alpha_2\tau^2 - \tfrac{1}{6}\alpha_3\tau^3 + \ldots..), \tag{1.4}$$

where $\tau \equiv t_0 - t$ and $\alpha_1 = \dot{R}_0/R_0$, $\alpha_2 = \ddot{R}_0/R_0$, etc., suffix 0 denoting quantities evaluated at the epoch t_0, and a dot indicating differentiation with respect to t.

In this notation we may write for the radial coordinate ρ of the source *at the epoch of emission*

$$\rho = c\tau\left[1 + \tfrac{1}{2}\alpha_1\tau + \left\{\tfrac{1}{6}(2\alpha_1^2 - \alpha_2) + \frac{c^2}{3a^2}\right\}\tau^2 + \ldots..\right]. \tag{1.5}$$

The red shift ratio δ $(=(\lambda_0-\lambda)/\lambda)$ of the radiation from a distant source, arising from the expansion of the universe, is given by

$$\delta=\frac{R_0}{R(t)}-1. \tag{1.6}$$

On inverting the series expansion of δ in terms of τ we obtain for τ in terms of δ

$$\tau=\left(\frac{1}{\alpha_1}\right)\delta+\left(\frac{\frac{1}{2}\alpha_2-\alpha_1^2}{\alpha_1^3}\right)\delta^2+O(\delta^3). \tag{1.7}$$

This will enable us to eliminate the parameter τ in favour of the observable δ when necessary.

2. *Angular diameter related to proper diameter.*—Consider in the first instance a distant radiating object which is ideally spherical according to local observers on it. Let its proper diameter according to such observers be d and let the radial coordinate of its centre be ρ, both these quantities applying to the epoch t of emission of the radiation that is registered by the observer at $\rho=0$ at epoch t_0. We do not assume that d and ρ are necessarily constant for the sphere. Let the angular diameter of the sphere as registered at $\rho=0$ be θ. Since our ultimate interest will be in galaxies and clusters of galaxies, or radio sources, whose intrinsic dimensions will be small compared with their distance from the observer, we shall regard θ as a small quantity of the first order.

Referring now to the metric (1.2) we may write to a first order approximation adequate for our purpose

$$d=\frac{R(t)\rho\theta}{R_0(1+\rho^2/a^2)}. \tag{2.1}$$

By this expression the proper diameter of the sphere at the instant of emission is related to its angular diameter registered by the observer.

Discussion

(i) *Application to galaxies or radio sources.*—It should be noted that the quantity d is not directly measurable by our observer at $\rho=0$, but d will have a statistical interpretation as a parameter of the cosmological model that we are trying to identify.

In the case of galaxies, because of their varying intrinsic size and the fact that they will present varying aspects to the observer, the proper 'diameter' d requires definition. In any observational test it is to be envisaged that a particular class of galaxies, elliptic say, would be selected. As a simple working procedure one might perform the angular measure in the direction of the greatest extension of the galaxy, as seen in projection, out to that distance on either side of the centre where the brightness fell to a certain fraction of that at the centre. There would thus be a mean value $\bar{\theta}$ for the angular 'diameter' of all galaxies of the class in a given neighbourhood, i.e. within certain limits of exhibited red shift or apparent magnitude. There would then be a corresponding mean value $\bar{d}$ for the proper extension of these galaxies at right angles to the line of sight. On a statistical basis this quantity would give a satisfactory measure of the intrinsic size of the average galaxy of the specified class *at the epoch of emission.*

For galaxies, therefore, our formula will be written in the form

$$\bar{d}=\frac{R(t)\rho\bar{\theta}}{R_0(1+\rho^2/a^2)}, \tag{2.2}$$

with the above interpretation of $\bar{d}$ and $\bar{\theta}$. In a restricted neighbourhood, at sufficient distance, both t and ρ will vary negligibly from galaxy to galaxy. For instance, if the neighbourhood be a whole cluster of galaxies we may with sufficient approximation take t and ρ to correspond to a galaxy at the centre of the cluster.

In the case of galaxies of the specified class selected from clusters selection effect could be minimized by restricting the choice to the nth brightest members of the cluster, where n is suitably chosen.

It is clear that in principle all this procedure could be applied to radio sources also. It seems probable, however, that most radio sources are too distant for their angular diameters to be measurable. It may also be difficult to detect clusters of radio sources for the purpose of applying our procedure against selection effect. Perhaps the most feasible test with radio sources will be to measure their angular separations as a function of their apparent magnitude (Section 5 (ii), (iv)).

It is important to observe that if an adequate sample of sources were chosen for measurement at all distances so that selection effect is eliminated, $\bar{d}$ could be expected to be independent of distance in the case of a steady-state universe. Thus, any systematic dependence of $\bar{d}$ on distance inferred from the observational data would confirm that the universe was evolutionary.

It is of course recognized that angular measurements applied to individual galaxies may present formidable difficulties at the great distances at which a systematic variation in intrinsic size could be expected to be detectable, and this seems certainly so in the case of radio sources. However, it must be emphasized that in the case of individual sources our formula is not primarily designed for this purpose, but rather to indicate a definite manner of variation of $\bar{\theta}$ with corresponding red shift δ (or apparent magnitude), via the parameters $R(t)$ and ρ, depending on the cosmological model adopted (Sections 4, 6). It may be possible to acquire reliable data for this purpose, from optical surveys at least, free of selection effect. Failing this, we may be able to determine the same information by studying the dimensions of the clusters themselves, as described below, or by the other observational tests introduced in Section 3.

(ii) *Application to clusters.*—To form a criterion for the intrinsic size of a cluster of galaxies (and possibly of radio sources) one might consider all the nth brightest members, 10th brightest say, of the cluster. These could be expected to be grouped randomly about the centre of the cluster. As a simple definition of the angular diameter of the cluster one might take the mean $\bar{\theta}$ of the angular separations of the five most widely separated pairs of the group, no single galaxy being used more than once for this purpose. There would then be a corresponding mean proper distance $\bar{d}$ separating these five pairs, measured at right angles to the line of sight, which to adequate approximation would be related to $\bar{\theta}$ in accordance with (2.2). In that formula t and ρ would again be associated with the centre of the cluster to this approximation. In accordance with the hypothesis stated in Section 1 the ρ coordinate of the centre of the cluster remains constant, independent of epoch.

The quantity $\bar{d}$, thus defined, could be taken to be a suitable statistical measure of the intrinsic size of the cluster. One would naturally choose n fairly large for statistical reasons, but small enough to permit the nth brightest galaxies of a statistically sufficient number of clusters in each neighbourhood of clusters to be observable. Once again, any systematic variation of $\bar{d}$ inferred from

observational data would point to an evolutionary universe. In addition, a definite cosmological model must exhibit a definite dependence of $\bar{\theta}$ on red shift or apparent magnitude.

3. *Angular separation of neighbouring objects related to their proper distance apart.*—Another important criterion for the cosmological problem besides the apparent size of the objects is their angular separation in space. As mentioned in Section 1 the angular separation of clusters has been treated by Florides and McCrea. In deriving a specific formula these authors took the view that any variation in the intrinsic size of the clusters with epoch could be justifiably neglected. In any case, they argued, such a variation would not in general prevent a distinction between an evolutionary universe and a steady-state one.

In the present paper, however, as in I and II, explicit allowance is made for possible systematic evolutionary trends, since there appears every likelihood that such important criteria will ultimately be observable. As an example we mention the recent exploratory analysis of the population of clusters by Just (5), which might be interpreted as tentative evidence for a statistical dependence of cluster size on epoch.

The size of an object has therefore been regarded as a function of epoch in general, and our working definition of intrinsic cluster size in Section 2 (ii) should reliably take account of any evolutionary variation. We shall also want to consider the separation of galaxies (or radio sources) which are gravitationally bound within the clusters as well as that of the clusters themselves, and the following analysis, although qualitatively similar to that of Florides and McCrea, will be rather different in detail.

Consider a radiating object A, whose radial coordinate is ρ at the epoch t of emission of the radiation that is registered by the observer O ($\rho=0$) at epoch t_0. Let there be another similar radiating object B which is A's nearest neighbour of the same class in the direction making angle ϕ, as measured by A, with OA. Let the angular separation of A and B as seen by O be Θ. Suppose also that the proper distance AB *at the time of emission* is D. (This symbol represented 'luminosity distance' in I but there need be no confusion here.) We shall regard D as small compared with OA and hence Θ as small of the first order.

Reference to the metric (1.2) then yields, to a first order approximation adequate for our purpose,

$$D \sin\phi = \frac{R(t)\rho\Theta}{R_0(1+\rho^2/a^2)}. \tag{3.1}$$

We now wish to find the mean value of Θ, supposing that D obeys a certain distribution law and the direction of B from A is arbitrary, all directions being equally likely. We first suppose that D is constant but ϕ and the plane OAB are arbitrary. Clearly, the mean value of Θ in this case will be

$$\Theta_{\text{mean}} = \frac{1}{4\pi}\int_0^{4\pi} \Theta \, d\omega, \tag{3.2}$$

where ω is the solid angle $2\pi(1-\cos\phi)$. Thus

$$\Theta_{\text{mean}} = \frac{R_0(1+\rho^2/a^2)}{R(t)\rho} \cdot \frac{\pi D}{4}. \tag{3.3}$$

If D is now supposed to follow its statistical distribution law, of mean $\bar{D}$, we

find that the mean angular separation of A from its neighbours at epoch t is $\bar{\Theta}$, given by

$$\frac{\pi \bar{D}}{4} = \frac{R(t)\rho\bar{\Theta}}{R_0(1+\rho^2/a^2)}. \tag{3.4}$$

It is to be noted that while $\bar{d}$ in Section 2 is a mean projected length measured at right angles to the line of sight, $\bar{D}$ on the other hand is the actual mean distance separating neighbouring objects. That is, in this case $\pi\bar{D}/4$ is the mean projected distance at right angles to the line of sight.

Discussion

(i) *Application to individual galaxies or radio sources.*—The substitution of a galaxy (or radio source) for object A, and the neighbours of this galaxy taken in turn for object B, requires care to guard against selection effect. It would be necessary, once again, to confine the examination to the nth brightest members of a cluster, with n suitably chosen and taking the galaxy A to be a central member of the group. Clusters of radio sources may not of course be detectable as such, but perhaps the problem of selection is not so acute in this case since radio sources are far less numerous than galaxies.

The manner of the dependence of $\bar{\Theta}$ on red shift or apparent magnitude, via the parameters $R(t)$ and ρ, is dealt with in Sections 5, 6, and in principle yields another method of identifying a cosmological model for the universe. It may also be possible to detect a progressive variation in the parameter $\bar{D}$, even although the sources may be gravitationally bound within the cluster. This of course would be incompatible with a steady-state universe.

(ii) *Application to clusters.*—In the case of clusters we require the mean value $\bar{\Theta}$ of the angular distances between the centre of a distant cluster A and the centres of its immediate neighbours. As a practical procedure one would define each cluster by its nth brightest members, for some n. For the angular distance between clusters A and B one would take the mean of the angular distances between each member of A and a corresponding member of B.

The parameters t and ρ in (3.4) will be associated with the centre of cluster A. According to the hypothesis in Section 1 the cluster centres take part in the expansion of the universe, the centres having constant coordinate ρ. It follows from (1.2) that, in an evolutionary universe in which clusters are conserved, the mean distance $\bar{D}$ between neighbouring clusters at epoch t will bear to their mean distance $\bar{D}_0$ at epoch t_0 (the present) the ratio

$$\frac{\bar{D}}{\bar{D}_0} = \frac{R(t)}{R_0}. \tag{3.5}$$

Thus, for clusters in a given distant neighbourhood of an evolutionary universe, (3.4) may be written

$$\frac{\pi \bar{D}_0}{4} = \frac{\rho\bar{\Theta}}{1+\rho^2/a^2}, \tag{3.6}$$

where ρ is the radial coordinate of the centre of the neighbourhood, and $\bar{D}_0$ is the mean distance separating clusters at epoch t_0—in particular those in the observer's own neighbourhood.

In a steady-state universe, on the other hand, the mean distance between clusters must remain statistically constant for all epochs, due to the steady formation

of new clusters in the space between receding clusters. Thus, in a steady-state universe,

$$\bar{D} = \bar{D}_0 = \text{constant}. \tag{3.7}$$

These results will form an important criterion for distinguishing between an evolutionary and a steady-state universe (Section 5).

4. *Observational tests involving angular diameters*

(i) *Angular diameter related to red shift.*—The formula relating the proper linear diameter (at the epoch of emission) to the observed angular diameter of galaxies, radio sources, and clusters, is (2.2), the interpretation of the symbols being given in Section 2 (i) and (ii) as appropriate.

We may now expand $R(t)$ and ρ in powers of the parameter τ according to (1.4) and (1.5), and in turn in terms of the red shift δ by means of (1.7). We thus obtain

$$\log_{10}(\delta\bar{\theta}) = \log_{10}\left(\frac{\alpha_1 \bar{d}}{c}\right) + 0{\cdot}217\left(3 - \frac{\alpha_2}{\alpha_1^2}\right)\delta + O(\delta^2). \tag{4.1}$$

The terms of order δ^2 are not specified because of their complication and the fact that they introduce two more unknowns whose evaluation by means of the observational data would be impossible. However, we are at liberty to formally allow for such a term in the analysis of the data by a least squares fitting, as described below.

The quantity α_2 is the important acceleration parameter of the expanding universe, α_1 being the Hubble parameter (1). It is evident that if $\bar{d}$ were statistically constant, independent of epoch or at any rate varying too slowly to be detectable, then the graph of $\log_{10}(\delta\bar{\theta})$ against δ would, to a first approximation, be a straight line of gradient $0{\cdot}217(3-\alpha_2/\alpha_1^2)$. In this case the value of the dimensionless parameter α_2/α_1^2 would be immediately determined from the data fitted by least squares to a relation of the form

$$\log_{10}(\delta\bar{\theta}) = A + B\delta + C\delta^2. \tag{4.2}$$

If α_2/α_1^2 turned out to be sufficiently different from unity (the value for a steady-state universe) then an evolutionary universe would be established.

It might be, however, that $\bar{d}$ varied sufficiently fast with epoch so that when expanded in series it contributed a term of significant magnitude to our first order term in δ. Thus, putting

$$\bar{d} = \bar{d}_0 - \dot{\bar{d}}_0\tau + O(\tau^2),$$

where

$$\dot{\bar{d}}_0 = \left\{\frac{d(\bar{d})}{dt}\right\}_0,$$

and using (1.7), we find

$$\log_{10}\left(\frac{\alpha_1 \bar{d}}{c}\right) = \log_{10}\left(\frac{\alpha_1 \bar{d}_0}{c}\right) - 0{\cdot}434\left(\frac{\dot{\bar{d}}_0}{\alpha_1 \bar{d}_0}\right)\delta + O(\delta^2). \tag{4.3}$$

Here $\bar{d}_0$ is the mean proper diameter for the sources at the present epoch t_0. The relation (4.1) now becomes

$$\log_{10}(\delta\bar{\theta}) = \log_{10}\left(\frac{\alpha_1 \bar{d}_0}{c}\right) + 0{\cdot}217\left(3 - \frac{\alpha_2}{\alpha_1^2} - \frac{2\dot{\bar{d}}_0}{\alpha_1 \bar{d}_0}\right)\delta + O(\delta^2). \tag{4.4}$$

By (4.2) and (4.4) the data would yield values for

$$A=\log_{10}\left(\frac{\alpha_1 d_0}{c}\right), \tag{4.5}$$

$$B=0\cdot217\left(3-\frac{\alpha_2}{\alpha_1^2}-\frac{2\dot{d}_0}{\alpha_1 d_0}\right). \tag{4.6}$$

By (4.5) a knowledge of either the Hubble parameter α_1 or of d_0 would allow a value of the other to be deduced, a result which might be of some interest.

By (4.6) we see that, although we might definitely *eliminate* the possibility of a steady-state universe by studying the gradient of the curve of $\log(\delta\bar{\theta})$ against δ, we could not *establish* it with certainty. The data might represent an evolutionary universe whose parameter α_2/α_1^2 we could not be sure of because of the term in $\dot{d}_0/d_0$. Fortunately, however, an independent means of deriving α_2/α_1^2 is provided by studying the angular separations of clusters, as shown in Section 5. The data on angular diameters will then lead via (4.6) to a value for $\dot{d}_0/d_0$, assuming that we know α_1; this will have its interpretation in terms of evolutionary trends in the galaxies.

(ii) *Angular diameter related to apparent magnitude.*—Since in the case of radio sources it is not at present possible to obtain their red shift directly, we give an alternative procedure involving apparent magnitude which may also be used for galaxies. Some preliminary measurements relating the angular diameters of galaxies, measured photoelectrically, to their apparent magnitudes have been reported by Baum (**6**).

The relation between the red shift δ and the apparent magnitude m of a distant source was derived in I and takes the form

$$m=M_0-5\log_{10}\alpha_1-5+5\log_{10}(c\delta)+1\cdot086\left(1+\frac{\alpha_2}{\alpha_1^2}+\kappa^*+\lambda^*\right)\delta+O(\delta^2). \tag{4.7}$$

Here m is the apparent magnitude as directly registered (apart from aperture or obscuration corrections). M_0 is the registered absolute magnitude of a standard local source of the same type. The term in κ^* represents the linearised red shift 'correction' which may be derived by a study of the standard spectrum. The term in λ^* allows for evolutionary change in the spectrum of the distant source compared with the standard source.

On eliminating δ between (4.4) and (4.7) we find the relation between $\bar{\theta}$ and m to be

$$\log_{10}\bar{\theta}=-1+\log_{10}d_0-0\cdot2(m-M_0)$$
$$+\frac{2\cdot17\alpha_1}{c}\left(4+\kappa^*+\lambda^*-\frac{2\dot{d}_0}{\alpha_1 d_0}\right)10^{0\cdot2(m-M_0)}+O\{10^{0\cdot4(m-M_0)}\}. \tag{4.8}$$

Therefore, if the observational data for $\bar{\theta}$ and m were fitted by least squares to a relation of the form

$$\log_{10}\bar{\theta}=E-\log_{10}y+Gy+Hy^2, \tag{4.9}$$

with $y=10^{0\cdot2(m-M_0)}$, M_0 assumed known, we should obtain values for the constants

$$E=-1+\log_{10}d_0, \tag{4.10}$$

$$G=\frac{2\cdot17\alpha_1}{c}\left(4+\kappa^*+\lambda^*-\frac{2\dot{d}_0}{\alpha_1 d_0}\right). \tag{4.11}$$

The relation (4.10) would give a value for the mean diameter d_0 of the sources, as defined in Section 2, at the present epoch. As regards the value for (4.11), we observe that in a steady-state universe, supposing that selection effect had been eliminated, we should have $\lambda^* = 0$ and $\dot{d}_0 = 0$. Therefore, given the values of α_1 and κ^*, we know what G should be for a steady-state universe. If G turned out to have a significantly different value then the steady-state model would be ruled out in favour of an evolutionary model. On the other hand a steady-state universe could not be definitely established on this test alone.

The disadvantages of using this method for galaxies are of course the fact that the parameters M_0 and κ^*, as well as α_1, have to be known, and that it does not distinguish between different evolutionary models. On the other hand apparent magnitudes are more easily determined than red shift.

5. *Observational tests involving angular separation*

(i) *Angular separation of galaxies related to red shift.*—The observed mean angular separation between a galaxy A and its neighbours is related to its properly measured mean distance from them at the epoch of emission by (3.4). This relation is formally the same as for angular diameters except for the extra constant factor on the left. Analogously to (4.4), therefore, we derive the following equation

$$\log_{10}(\delta\bar{\Theta}) = \log_{10}\left(\frac{\pi\alpha_1 D_0}{4c}\right) + 0{\cdot}217\left(3 - \frac{\alpha_2}{\alpha_1^2} - \frac{2\dot{D}_0}{\alpha_1 D_0}\right)\delta + O(\delta^2). \qquad (5{\cdot}1)$$

Here δ is the red shift of the galaxy A, D_0 is the mean proper distance separating it from its neighbours at the present epoch t_0, while $\dot{D}_0$ is the present rate of increase of this distance with epoch ($\dot{D}_0$ may of course be negative).

The $\bar{\Theta} - \delta$ observational data would thus provide values for the constants

$$A' = \log_{10}\left(\frac{\pi\alpha_1 D_0}{4c}\right) \qquad (5{\cdot}2)$$

$$B' = 0{\cdot}217\left(3 - \frac{\alpha_2}{\alpha_1^2} - \frac{2\dot{D}_0}{\alpha_1 D_0}\right). \qquad (5{\cdot}3)$$

Eqn. (5.2) allows us to deduce D_0 if we know α_1 and vice versa. In a steady-state universe $\dot{D}_0$ will be zero statistically, in the sense that galaxies of a given class must always be found at the same average distance apart if the average for a region (epoch) be taken over a sufficiently large number of clusters belonging to that region (epoch). Consequently, since α_2/α_1^2 is unity for this case, a coefficient of δ found from the data to be significantly different from 0·434 would rule out the steady-state model, and establish that the universe was evolutionary.

In an evolutionary universe $\dot{D}_0$ may or may not be zero, even if the galaxies are gravitationally bound in clusters. The assumption that $\dot{D}_0$ is zero would permit the value of the important parameter α_2/α_1^2 to be deduced. Once again, if we do not make this assumption, then a steady-state universe could not be definitely established and the value of α_2/α_1^2 must be derived from the corresponding data on the separation of clusters, as described in (iii). Substitution of this value for α_2/α_1^2 into (5.3) would then provide a value for $\dot{D}_0/D_0$, given α_1. This would represent valuable information as to evolutionary trends within the clusters.

(ii) *Angular separation of galaxies or radio sources related to apparent magnitude.*—To cater specially for radio sources and as an alternative treatment for

galaxies we may derive a relation, analogous to (4.8), connecting $\bar{\Theta}$ to apparent magnitude m, in the form

$$\log_{10}\bar{\Theta} = -1 + \log_{10}\left(\frac{\pi \bar{D}_0}{4}\right) - 0{\cdot}2\,(m - M_0)$$
$$+ \frac{2{\cdot}17\alpha_1}{c}\left(4 + \kappa^{\bullet} + \lambda^{\bullet} - \frac{2\dot{\bar{D}}_0}{\alpha_1 \bar{D}_0}\right) 10^{0{\cdot}2(m-M_0)} + O\{10^{0{\cdot}4(m-M_0)}\}. \qquad (5{\cdot}4)$$

Discriminative comments apply to this relation analogous to those made in the case of (4.8), where we now read 'distance of separation' for 'diameter'.

(iii) *Angular separation of clusters related to red shift.*—In the case of clusters we have shown in Section 3 (ii) that, because of our hypothesis that the cluster centres have constant reference coordinates ρ, θ, ϕ and thus take part in the expansion of the universe, the formula (3.4) is consequently qualified by (3.5), leading to (3.6), in the case of an evolutionary universe, whereas for a steady-state universe (3.4) is qualified only by (3.7).

Employing the expansions (1.5) and (1.7) we therefore find that in an evolutionary universe the mean angular separation $\bar{\Theta}$, defined as in Section 3 (ii), of a cluster A from its neighbours is related to the red shift exhibited by a central galaxy of A by the equation

$$\log_{10}(\delta\bar{\Theta}) = \log_{10}\left(\frac{\pi\alpha_1 \bar{D}_0}{4c}\right) + 0{\cdot}217\left(1 - \frac{\alpha_2}{\alpha_1^2}\right)\delta + O(\delta^2), \qquad (5{\cdot}5)$$

where $\bar{D}_0$ would be the mean distance separating A from its neighbours at the present epoch t_0.

On the other hand, in a steady-state universe (3.4) combined with (3.7) yield us (5.1) again, except that we have now to put $\dot{\bar{D}}_0 = 0$ and $\alpha_2/\alpha_1^2 = 1$. Thus, for the steady-state model

$$\log_{10}(\delta\bar{\Theta}) = \log_{10}\left(\frac{\pi\alpha_1 \bar{D}_0}{4c}\right) + 0{\cdot}434\delta + O(\delta^2), \qquad (5{\cdot}6)$$

where $\bar{D}_0$ is the mean distance separating a cluster from its neighbours at the present epoch and at any other epoch.

On fitting the $\bar{\Theta} - \delta$ data for clusters at successive distances from the observer to a relation of the form

$$\log_{10}(\delta\bar{\Theta}) = A'' + B''\delta + C''\delta^2 \qquad (5{\cdot}7)$$

we would find values for the constants

$$A'' = \log_{10}\left(\frac{\pi\alpha_1 \bar{D}_0}{4c}\right), \qquad (5{\cdot}8)$$

and
$$B'' = \begin{cases} 0{\cdot}217\left(1 - \dfrac{\alpha_2}{\alpha_1^2}\right) & \text{in an evolutionary universe} \\ 0{\cdot}434 & \text{in a steady-state universe.} \end{cases} \qquad (5{\cdot}9)$$

From (5.8) given the Hubble parameter α_1 we may calculate $\bar{D}_0$ for either kind of universe.

The eqn. (5.9) is of the utmost value since it directly provides the acceleration parameter α_2/α_1^2, whose sign determines whether the expansion of the universe is speeding up or slowing down at the present epoch. Unless B'' turned out to be close to 0·434 the universe would be established as evolutionary. It is to be noted, however, that even if B'' was near to 0·434 it might signify an evolutionary universe

for which α_2/α_1^2 was equal to -1. To remove ambiguity in this case one could appeal to the other tests described in this paper, or of course to the tests described in I and II.

(iv) *Angular separation of clusters related to apparent magnitude.*—Assuming that radio sources occur mainly within the clusters of galaxies, in accordance with the collision hypothesis for their principal origin, it may be useful to provide a formula for the angular separation of clusters in terms of the apparent magnitude of representative radio sources within them. As mentioned in Section 2 (i), it may be difficult to identify clusters of radio sources, but if radio sources occur only in clusters this would not matter. The formula will in any case be suitable as an alternative treatment for clusters of galaxies observed optically.

In the case of an evolutionary universe it is clear that the required formula will simply be (5.4), where $\bar{D}_0$ is now the mean distance between cluster centres at the present epoch and, in virtue of (3.5), $\dot{\bar{D}}_0=\alpha_1\bar{D}_0$. Hence

$$\log_{10}\bar{\Theta}=-1+\log_{10}\left(\frac{\pi\bar{D}_0}{4}\right)-0{\cdot}2(m-M_0)$$
$$+\frac{2{\cdot}17\alpha_1}{c}(2+\kappa^*+\lambda^*)10^{0\cdot2(m-M_0)}+O\{10^{0\cdot4(m-M_0)}\}. \quad (5.10)$$

For a steady-state universe we have to put $\dot{\bar{D}}_0=0$. In addition $\lambda^*=0$, since statistically the sources will be of strength independent of epoch. In this case, therefore,

$$\log_{10}\bar{\Theta}=-1+\log_{10}\left(\frac{\pi\bar{D}_0}{4}\right)-0{\cdot}2\,(m-M_0)$$
$$+\frac{2{\cdot}17\alpha_1}{c}(4+\kappa^*)\,10^{0\cdot2(m-M_0)}+O\{10^{0\cdot4(m-M_0)}\}. \quad (5.11)$$

The fitting of the data to a relation of the form

$$\log_{10}\bar{\Theta}=E'-\log_{10}y+G'y+H'y^2, \quad (5.12)$$

where $y=10^{0\cdot2(m-M_0)}$, would present us with values for the constants

$$E'=-1+\log_{10}\left(\frac{\pi\bar{D}_0}{4}\right), \quad (5.13)$$

$$G'=\begin{cases}\dfrac{2{\cdot}17\alpha_1}{c}(2+\kappa^*+\lambda^*) & \text{in an evolutionary universe}\\[2ex] \dfrac{2{\cdot}17\alpha_1}{c}(4+\kappa^*) & \text{in a steady-state universe.}\end{cases} \quad (5.14)$$

Eqn. (5.13) gives us the mean distance between cluster centres at the present epoch. Eqn. (5.14) indicates that there is a definite value for G' in a steady-state universe which is ascertainable since κ^* and α_1 will be known. Coincidence apart, therefore, the two kinds of universe should be readily distinguishable. This would be especially so if λ^* is negative for an evolutionary universe (average power of the sources diminishing as the epoch advances).

(v) *The test of Florides and McCrea.*—As a measure of the congestion of the universe in a given region, Florides and McCrea (**3**) suggest dividing the mean angular separation of the clusters in that region by the mean angular diameter of these clusters. Except in a special circumstance this ratio will be a function of distance in an evolutionary universe, but constant in a steady-state universe, so that this constitutes a valuable means of distinguishing between these two kinds

of universe. According to the analysis developed in this paper their test may be expressed as follows.

For clusters in a given neighbourhood of an evolutionary universe we get, dividing (3.4) by (2.2),

$$\frac{\bar{\Theta}}{\bar{\theta}} = \frac{\pi \bar{D}}{4d}. \tag{5.15}$$

(Our practical procedure for finding $\bar{\theta}$ and $\bar{\Theta}$, as defined in Sections 2 and 3, is rather different from that of Florides and McCrea but this need not concern us here.) Also, the mean distance $\bar{D}$ separating a cluster from its neighbours at the epoch of emission is related in accordance with (3.5) to the mean distance separating them at the present epoch t_0. Hence

$$\frac{\bar{\Theta}}{\bar{\theta}} = \frac{\pi \bar{D}_0}{4d} \cdot \frac{R(t)}{R_0}. \tag{5.16}$$

It follows that in an evolutionary universe the ratio of the mean angular separation of a cluster A from its neighbours to the angular diameter of A may be written, observing (1.6),

$$\frac{\bar{\Theta}}{\bar{\theta}} = \frac{\pi \bar{D}_0}{4d} \cdot \frac{1}{1+\delta}, \tag{5.17}$$

where δ is the red shift exhibited by the central region of A.

On the other hand, for a steady-state universe dividing (3.4) by (2.2) and putting $\bar{D} = \bar{D}_0$, $d = d_0$, leads to

$$\frac{\bar{\Theta}}{\bar{\theta}} = \frac{\pi \bar{D}_0}{4d_0}. \tag{5.18}$$

It follows that unless a possible variation of d in (5.17) exactly masks the trend in the factor $1/(1+\delta)$ the ratio of the observables must vary systematically for an evolutionary universe while remaining constant for a steady-state universe.

Should the ratio turn out to be in fact constant any ambiguity could only be settled by appealing to another test. In any case, recourse to other tests would be necessary even if the ratio were not constant, if we wanted to distinguish between different evolutionary models by determining kinematic parameters such as α_2/α_1^2.

6. *Some exact observable relations for a steady-state universe*

(i) *Introduction.*—For the steady-state model the metric factor $R(t)$ is equal to $e^{t/T}$, where T is a constant which is evidently the reciprocal of the Hubble parameter α_1 for the model. In addition the curvature of space is zero so that $1/a^2 = 0$. It is therefore possible to give exact expressions for some of our relations between observables in this case, always bearing in mind that our formulae are subject to the restriction that the mean angular diameter $\bar{\theta}$, or the mean angular separation $\bar{\Theta}$, of the objects is a *small* quantity, so that an approximation in this respect has already been made.

It will, unfortunately, not be possible to express angular diameter and angular separation in terms of apparent magnitude by an exact relation. This is because the exact red shift 'correction' to registered apparent magnitudes, discussed in I, contains δ explicitly, making necessary a measurement of red shift as well as apparent magnitude. If we wished to confine distance measurements to apparent magnitude only, we must therefore fit the data to the approximate relations already

dealt with. In these, of course, δ has been eliminated in terms of apparent magnitude by a first order approximation.

Referring to (1·6) we see that in the steady-state model the red shift δ is related to the epoch t of emission of the radiation by the equation

$$\delta = e^{(t_0 - t)/T} - 1, \tag{6.1}$$

where as usual t_0 is the epoch of its reception.

Also, the null geodesic followed by the radiation between its source, of radial coordinate ρ at the moment of emission, and the observer at $\rho = 0$ has the equation, in the general case,

$$\int_0^\rho \frac{d\rho}{1 + \rho^2/a^2} = c R_0 \int_t^{t_0} \frac{dt}{R(t)}. \tag{6.2}$$

In the case of the steady-state model this leads to

$$\rho = c\,T\,(e^{(t_0 - t)/T} - 1),$$

or

$$\rho = cT\delta, \tag{6.3}$$

by (6.1).

(ii) *Angular diameter related to red shift.*—By (2.2) and (6.3) we obtain for a steady-state universe the exact relation between the mean angular diameter of an object and its red shift in the form

$$\bar{\theta} = \frac{d_0(1 + \delta)}{cT\delta}, \tag{6.4}$$

where we have put $\bar{d} = d_0$ as $\bar{d}$ is statistically independent of epoch in this model.

We see that in a steady-state universe the mean angular diameters of galaxies, radio sources, and clusters of these objects will decrease monotonically as their exhibited red shift increases, towards a definite lower limit d_0/cT at the observational horizon of the universe ($\delta = \infty$). This limit is a calculable quantity if we ascertain d_0 by local observations and know the value of the Hubble parameter $1/T$.

The result (6.4) has been derived and discussed by the author previously (**7**), when an estimate was given for the limiting angular diameter of a standard galaxy in a steady-state universe. Since then the Hubble constant has been revised, the present estimated value for its reciprocal being 13×10^9 yrs. Taking this value for T we find that the limiting value of $\bar{\theta}$, corresponding to $d_0 = 30{,}000$ lt. yrs. say, would be approximately 0·5 seconds of arc. For clusters, taking for d_0 the rough figure of 5×10^6 lt. yrs., the lower limit would be nearly 80 seconds of arc.

Formula (6.4) has been discussed more recently by Hoyle (**8**). He calculates that the limiting value of $\bar{\theta}$ (our notation) for a radio source, of a standard intrinsic diameter equal to that of the source in Cygnus, would be approximately 4 seconds of arc. For comparison Hoyle has given an interesting discussion of the analogous situation in the Einstein–de Sitter universe, a well known *evolutionary* model. He finds that in this model, neglecting any progressive variation of the intrinsic diameter, $\bar{\theta}$ would decrease to a minimum at $\delta = 5/4$, and thereafter steadily increase. For the standard radio source described above he finds that the minimum value of $\bar{\theta}$ would be approximately 15 seconds of arc.

It may be shown, however, that this behaviour is not general for the evolutionary models, even in the unlikely event that $\bar{d}$ remains constant. For instance, in a Milne universe $\bar{\theta}$ would decrease monotonically to the value $2\bar{d}\alpha_1/c$ at the observational horizon, α_1 being the Hubble constant. This limit is incidentally

twice that of the steady-state model. The behaviour of $\bar{\theta}$ in the evolutionary models depends on the metric factor $R(t)$, the curvature of space, and the variability of $\bar{d}$, so that no useful general criterion is available for these models in an exact form.

The most positive conclusion that we can make, therefore, is that since the exact form of dependence of $\bar{\theta}$ on δ in the steady-state model is likely to be unique, the strict behaviour (within statistical error) of the observables in accordance with (6·4) would represent powerful evidence in support of the steady-state model. On the other hand, any disagreement of the data with this relation would rule out such a universe.

(iii) *Angular separation related to red shift.*—Formula (3.4) relates the mean proper distance separating neighbouring galaxies, radio sources, or clusters, at the epoch of emission, to their observed mean angular separation. For the steady-state model, putting $\bar{D}=\bar{D}_0$, we therefore find

$$\frac{\pi\bar{D}_0}{4}=\rho\, e^{(t-t_0)/T}\bar{\Theta}.$$

Whence
$$\bar{\Theta}=\frac{\frac{\pi\bar{D}_0}{4}(1+\delta)}{cT\delta}. \tag{6.5}$$

Thus, in the same way as for angular diameters, the mean angular separations of these objects decrease monotonically as the distance increases, to the lower limit $\pi\bar{D}_0/4cT$. Once again this is a calculable quantity. For galaxies of a certain order of brightness, separated within the clusters by a mean distance of 10^6 lt. yrs. say, the limiting angular separation would be approximately 12 seconds of arc. For whole clusters separated by a mean distance of 2×10^7 lt. yrs. the lower limit of $\bar{\Theta}$ would be over 4 minutes of arc.

In the case of the *evolutionary* models no general statement of an exact nature can be made about the mean angular separation of galaxies within the clusters, not least because of the possible dependence of their mean distance $\bar{D}$ on epoch. For the clusters themselves, however, it is possible to be more specific. Since we have assumed that the cluster centres follow the expansion of the universe, the formula (3.6) applies. The author has found that the possible types of behaviour of the mean angular separation $\bar{\Theta}$ of cluster centres may be categorised as follows.

(*a*) Curvature of space positive. In this case, if space is spherical so that antipodal points are not identified, then it is possible for $\bar{\Theta}$ to steadily decrease to a minimum value of $\pi\bar{D}_0/4R_0$ (with our usual meanings for the symbols) occurring at $\rho=2R_0$, and thereafter to steadily increase. However, the limiting value of $\bar{\Theta}$ depends on the location of the observational horizon in the model, which in turn depends on the function $R(t)$ in (1.2). If space is elliptic then antipodal points are identified, $\bar{\Theta}$ would be monotonic decreasing and the limiting value of $\bar{\Theta}$ would be $\geqslant\pi\bar{D}_0/4R_0$ depending on the location of the horizon.

(*b*) Curvature of space equal to zero. For these models $\bar{\Theta}$ must steadily decrease to a limiting value which is $\geqslant 0$ according to the location of the horizon, whose coordinate ρ will be $\leqslant\infty$.

(*c*) Curvature of space negative. In this case $\bar{\Theta}$ must steadily decrease to a lower limit $\geqslant 0$, occurring at the observational horizon of coordinate $\rho\leqslant 2R_0$.

It may easily be verified that in the Einstein–de Sitter universe, which belongs to category (*b*), the limit for $\bar{\Theta}$ is the non zero value $\pi\bar{D}_0\alpha_1/8c$. This is half the

steady-state value. The Milne universe is of category (c) and in this case the lower limit for $\bar{\Theta}$ is zero.

We may assert, therefore, that the formula (6.5) represents another powerful criterion for the steady-state model, especially in its application to the angular separation of clusters. In particular, if beyond a certain distance the angular separations of neighbouring sources, or clusters of sources, were observed to be consistently less than the respective limits for the steady-state model then that universe would be eliminated.

(iv) *The ratio of angular separation to angular diameter.*—The expression for this ratio, namely

$$\frac{\bar{\Theta}}{\bar{\theta}} = \frac{\pi \bar{D}_0}{4 d_0}, \tag{6.6}$$

which was derived and discussed in Section 5(v), is another exact result for the steady-state model in the sense described in (i).

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References

(1) W. Davidson, *M.N.*, **119**, 54, 1959.
(2) W. Davidson, *M.N.*, **119**, 665, 1959.
(3) P. S. Florides and W. H. McCrea, *Zs. f. Ap.*, **48**, 52, 1959.
(4) W. H. McCrea, see J. Neyman and E. L. Scott, *J. R. Statist. Soc.* B, **20**, 1, 1958.
(5) K. Just, *Ap. J.*, **129**, 268, 1959.
(6) W. A. Baum, *Astron. J.*, **58**, 211, 1953.
(7) W. Davidson, Ph.D. Thesis, Part I, 76, University of London, 1958.
(8) F. Hoyle, *Paris Symposium on Radio Astronomy*, 1958, Stanford University Press, 1959, p. 529.

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